World Engineering & Applied Sciences Journal 9 (4): 105-106, 2018
ISSN 2079-2204

DOI: 10.5829/idosi.weasj.2018.105.106

Local Gauss-Bonnet Theorem

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Abstract: Horndeski showed that in arbitrary 2-Riemannian space the Gaussian curvature is an exact divergence, then we employ this fact to exhibit an elementary proof of the local Gauss-Bonnet theorem.

Key words: Gauss-Bonnet theorem · Differential geometry of surfaces · Green theorem

INTRODUCTION

Horndeski [1] proved that the Gaussian curvature of a surface [2, 3] is an exact divergence:

$$K = (\delta^{ab}_{cd}\ Ac\ A^{d}{}_{;a})_{;b},\quad \delta^{ab}_{cd} = \delta^{a}_{c}\delta^{b}_{a} - \delta^{a}_{a}\delta^{b}_{c}, \tag{1}$$

where A^r is an arbitrary unitary vector field.

In Sec. 2 we employ (1) to give a simple proof of the important local Gauss-Bonnet theorem [2, 3]:

$$\int\int_{s} K\ dS + \int_{c}\ \sigma\ ds = 2\pi, \tag{2}$$

such that C is any smooth closed contour on the 2-surface, S is its interior, with s and σ the arc-length and the geodesic curvature of C, respectively.

Local Gauss-Bonnet Theorem: The expression (1) leads to the analysis of A^r as a previous step to the proof of (2). On C the vector A^j can be written as:

$$A^{i} = pt^{i} + qn^{i},\quad t^{r}n_{r} = 0,\ p^{2} + q^{2} = 1, \tag{3}$$

in terms of the unitary tangent and normal vectors of C, verifying the Frenet formulae of the curve C relative to the surface:

$$\frac{\delta}{\delta s}t^{r} = \sigma n^{r},\quad \frac{\delta}{\delta s}n^{r} = -\sigma t^{r}, \tag{4}$$

where $\frac{\delta}{\delta s}$ is the absolute derivative on C. Hence from (3) and (4) are immediate the relations:

$$t_{i}\frac{\delta A^{i}}{\delta_{s}} = \frac{dp}{ds} - q\sigma,\quad n_{i}\frac{\delta A^{i}}{ds} + p\sigma, \tag{5}$$

besides, the metric tensor of the surface can be written on C in the form:

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$$g_{ab} = t_a t_b + n_a n_b \quad \therefore \quad \delta^a_b = t^a t_b + n^a n_b. \tag{6}$$

On the other hand, from (1) and the Green theorem [2, 3]:

$$\iint_s K\ dS = -\int_c n_b\ \delta^{ab}_{cd}\ A^c\ A^d{}_{;a}\, ds, \tag{7}$$

then now we study the integrand of (7) on C:

$$n_b\ \delta^{ab}_{cd}\ A^c\ A^d{}_{;a} \overset{(2)}{=} n_b \left(A^a\ A^b{}_{;a} - A^b\ A^a{}_{;a}\right) \overset{(3)}{=} n_b\ (p\ t^a + q\ n^a)\, A^b{}_{;a} - q\ A^a{}_{;a}\ ,$$

$$\overset{(6)}{=} p\, n_b \frac{\delta A^b}{\delta s} + q\ (\delta^a_b - t^a\ t_b)\ A^b{}_{;a} - q\ A^a{}_{;a} \overset{(3),(5)}{=} p\ \frac{dq}{ds} - q\ \frac{dp}{ds} + \sigma = p^2\ \frac{d}{ds}\left(\frac{q}{p}\right) + \sigma = \frac{d\varphi}{ds} + \sigma, \tag{8}$$

such that φ is the angle between A^r and t^r, that is $p = \cos\ \varphi$ and $q = \sin\ \varphi$.

Finally, the application of (8) in (7) implies the local Gauss-Bonnet theorem expressed in (2), where the sense the sense of integration on C is counterclockwise.

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