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Some Identities Involving Harmonic Sums

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Abstract: We exhibit elementary proofs of two identities involving harmonic numbers.

Key words: Harmonic sums · Riemann zeta function

INTRODUCTION

Here we give elementary deductions of the following identities:

$$\frac{H_k}{k}=\frac{1}{2}\left(H_k^2-H_{k-1}^2+\frac{1}{k^2}\right), \tag{1}$$

and [1]: $k \geq 1$

$$\frac{H_k H_{k-1}}{k}=\frac{1}{3}\left(H_k^3-H_{k-1}^3+\frac{1}{k^3}\right), \tag{2}$$

where $H_0 = 0$, $\sum_{r=1}^{n}\frac{1}{r}$, $n \geq 1$ are harmonic numbers [2, 3].

Harmonic Sums: It is clear that $H_{k-1}=H_k-\frac{1}{k}$, , then:

$$H_k^2-H_{k-2}^2=(H_k-H_{k-1})(H_k+H_{k-1})=\frac{1}{k}\left(2H_k-\frac{1}{k}\right)=2\frac{H_k}{k}-\frac{1}{k^2},$$

thus (1) is immediate; from (1) is simple to obtain the relation [1, 4, 5]:

$$\sum_{r=1}^{n}\frac{H_k}{k}=\frac{1}{2}\left(H_n^2+H_n^{(2)}\right), \tag{3}$$

involving the harmonic sum $H_n^{(2)}=\sum_{r=1}^{n}\frac{1}{r^2}.$

On the other hand, we have the following property of the Riemann zeta function [1]:

$$\zeta(2)=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{2^n}\left(H_n^2+H_n^{(2)}\right), \tag{4}$$

then from (1) and (4):

$$\zeta(2)=\sum_{k=1}^{\infty}\frac{H_k}{k}\sum_{n=k}^{\infty}\frac{1}{2^n}=\sum_{k=1}^{\infty}\frac{H_k}{k}\left(\sum_{k=0}^{\infty}\frac{1}{2^n}-\sum_{j=0}^{k-1}\frac{1}{2^j}\right)=2\sum_{k=1}^{\infty}\frac{H_k}{k.2^k}, \tag{5}$$

in according with Choi-Srivastava [6].

Similarly:

$$H_k^3-H_{k-1}^3=(H_k-H_{k-1})(H_k^2+H_kH_{k-1}+H_{k-1}^2)=\frac{1}{k}\left(H_k^2+H_kH_{k-1}+H_k^2-2\frac{H_k}{k}+\frac{1}{k^2}\right),=\frac{1}{\mathrm{k}^3}+\frac{1}{k}\left(2H_k^2-2\frac{H_k}{k}+H_kH_{k-1}\right)=\frac{1}{k^3}+2\frac{H_k}{k}\left(H_k-\frac{1}{k}\right)+\frac{H_kH_{k-1}}{k},$$

implying the identity (2); besides, from (2) we obtain the expression [1]:

$$\sum_{r=1}^{n}\frac{H_kH_{k-1}}{k}=\frac{1}{3}\left(H_n^3+H_n^{(3)}\right), \qquad H_n^{(3)}=\sum_{r=1}^{n}\frac{1}{r^3}, \tag{6}$$

The relations (1) and (2) can be unified in the form:

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$$\frac{1}{k}H_k\left[H_{k-(m-2)}\right]H^{m-2} = \frac{1}{m}\left[H_k^m - H_{k-1}^m + \frac{(-1)^m}{k^m}\right], \text{ m=2,3.} \quad (7)$$

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