World Engineering & Applied Sciences Journal 9 (4): 93-100, 2018
ISSN 2079-2204

DOI: 10.5829/idosi.weasj.2018.93.100

Matrix Approach to Point Symmetries of Lagrangians

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Abstract: We give an elementary exposition of a matrix approach to deduce the infinitesimal point symmetries of Lagrangians

Key words: Gauge identities · Singular Lagrangians · Local symmetries

INTRODUCTION

We consider a physical system where the parameters $q_1, q_2, \ldots, q_n$ are its generalized coordinates, that is, there are n degrees of freedom. The action:

$$S = \int_{t_1}^{t_2} L(q,\dot{q},t)dt, \quad \dot{q} = \frac{dq}{dt}, \tag{1}$$

is fundamental in the dynamical evolution of the system. We can change to new coordinates via the local transformations:

$$\tilde{t} = t + \varepsilon\alpha_0(t), \quad \tilde{q}_i = q_i + \varepsilon\alpha_i(q,t), \; I = 1, 2, \ldots, n \tag{2}$$

where ε is an infinitesimal parameter, thus the action takes the value:

$$\tilde{S} = \int_{\tilde{t}_1}^{\tilde{t}_2} L(\tilde{q}, \frac{d\tilde{q}}{d\tilde{t}}, \tilde{t})d\tilde{t}. \tag{3}$$

If $\delta S = \delta\tilde{S}$ to first order in ε, then we say that the action is invariant under the transformations (2), that is, (2) are local symmetries of the Euler-Lagrange equations of motion.

The principal aim of this work is to show a matrix technique to investigate the existence of point symmetries for a given action and to realize the explicit construction of the functions α_r. The Sec. 2 considers the particular (but important) case of $\alpha_0 = 0$ with $L(q,\dot{q})$, that is, when the time remains intact and L does not have explicit dependence in t. This particular situation is studied employing a matrix procedure [1-3] to obtain the so-called gauge identities, from which the coordinate transformations (local symmetries) leaving the action invariant, can be extracted. Local symmetries of the action are not always easily detected; it is however crucial to unravel them since their knowledge is required for the quantization [2] of such singular systems. We make applications to Lagrangians studied by several authors [2, 4-6].

Matrix Algorithm: Here we study the special transformations:

$$\tilde{t} = t, \quad \tilde{q}_i = q_i + \varepsilon\alpha_i(q,t), \tag{4}$$

when t is an ignorable variable, that is, the Lagrangian only depends of q_j and $\dot{q}_j$; then the variation of the action (1) is:

$$\delta S = -\int_{t_1}^{t_2} dt \;\; E_i^{(0)} \;\; \delta q_i = -\int_{t_1}^{t_2} dt \overline{E}^{(0)} . \delta\vec{q}, \quad \delta q_i(t_j) = 0, j = 1,2 \tag{5}$$

with the zeroth level Euler derivatives:

$$E_i^{(0)}(q,\dot{q},\ddot{q},t) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}, \quad i = 1,2,\ldots,n \tag{6}$$

thus $\delta S = 0$ if:

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$$\delta\vec{q}.\vec{E}^{(0)} = -\frac{d(\varepsilon Q)}{dt}, \quad Q(t_2) = Q(t_1) = 0, \tag{7}$$

without the participation of the equations of motion $E_i^{(0)} = 0$.

It is suitable to realize a splitting into (6) to show explicitly the presence of the accelerations:

$$\vec{E}^{(0)} = \left(E_i^{(0)}\right) = W^{(0)}.\ddot{\vec{q}} + \vec{K}^{(0)}, \quad \ddot{\vec{q}} = (\ddot{q}_j)_{nx1}, \tag{8}$$

with the Hessian matrix:

$$W^{(0)}(q,\dot{q}) = (W_{ij}{}^{(0)}) = \left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \ddot{q}_j}\right)_{nxn} \tag{9}$$

and the zeroth level vector:

$$\vec{K}^{(0)}(q,\dot{q}) = (K_i^{(0)}) = \left(\frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}\dot{q}_j - \frac{\partial L}{\partial q_i}\right)_{nx1}. \tag{10}$$

where the Dedekind (1868)-Einstein [7, 8] summation convention is used for repeated indices.

If p_c is the rank [9, 10] of $W^{(0)}$, then there exist $N_0 = n - p_0$ constraints, which on Lagrangian level present themselves as relations between coordinates q_i and velocities $\dot{q}_i$. We note that $N_0 =$ Nullity $W^{(0)}$ therefore there exist N_0 independent left zero mode eigenvectors $\vec{\varpi}^{(0,k)}$ of $W^{(0)}$:

$$\vec{\varpi}^{(0,k)}.W^{(0)} = 0, \quad r = 1,\ldots,N_0 \tag{11}$$

and we construct the quantities:

$$\Phi^{(0,r)} = \vec{\varpi}^{(0,r)}(q,\dot{q}).\,\vec{E}^{(0)}(q,\dot{q},\ddot{q}) = \vec{\varpi}^{(0,r)},\vec{K}^{(0)}, \tag{12}$$

which only depend on the coordinates and velocities and vanish on the subspace of physical trajectories:

$$\Phi^{(0,r)}(q,\dot{q}) = 0, r = 1,\ldots,N_0 \quad \text{if} \quad \vec{E}^{(0)} = \vec{0}, \tag{13}$$

we refer to them as the zero generation constraints.

Not all of the functions (12) may, however, be linearly independent. In this case one can find d_0 linear combinations of the zero mode eigenvectors:

$$\vec{v}^{(0,m_0)} = b_r{}^{(m0)}\vec{\varpi}^{(0,r)}, \quad r = 1,\ldots,N_0, \quad m_0 = 1,\ldots,d_0 \tag{14}$$

such that we obtain identically:

$$G^{(0,m_0)}(q,\dot{q}) = \vec{v}^{(0,m_0)}.\vec{E}^{(0)} = \vec{v}^{(0,m_0)}.\vec{K}^{(0)} = 0,$$

Gauge identities. (15)

In (14) we have d_0 linearly independent vectors into the subspace Kernel $W^{(0)}$ of dimension N_0, then we construct $\bar{N}_0 = N_0 - d_0$ vectors $\vec{u}^{(0,\bar{m}_0)}(q,\dot{q})$ to establish a base for this subspace and thus to introduce the quantities:

$$\varphi^{(0,\bar{m}_0)} = \vec{u}^{(0,\bar{m}_0)}.\vec{E}^{(0)} = \vec{u}^{(0)} = \vec{u}^{(0,\bar{m}_0)}.\vec{K}^{(0)}, \quad \bar{m}_0 = 1,\ldots,\bar{N}_0, \tag{16}$$

which vanish if $\vec{E}^{(0)} = \vec{0}$:

$$\varphi^{(0,\bar{m}_0)}(q,\dot{q}) = 0, \quad \textit{Genuine constraints}. \tag{17}$$

The identities (15) imply that any variation of the form:

$$\delta\vec{q} = \varepsilon\beta m_0(t)\vec{v}(0,m_0), \quad m_0 = 1,\ldots,d_0 \tag{18}$$

verifies (7) with $Q = 0$ and it will leave the action invariant.

Next we look for possible additional constraints by searching for further functions of the coordinates and velocities which vanish on the subspace of physical paths. To this effect we employ the following vector constructed from $\vec{E}^{(0)}$ and the time derivative of the non-trivial quantities (16):

$$\vec{E}^{(1)} = \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d}{dt}\varphi^{(0,1)} \\ \vdots \\ \frac{d}{dt}\varphi(0,\bar{N}_0) \end{pmatrix}_{(n+\bar{N}_0)x1} = W^{(1)}.\ddot{\vec{q}} + \vec{K}^{(1)}(q,\dot{q}), \tag{19}$$

a splitting similar to (8), where $W^{(1)}(q,\dot{q})$ is now the 'level 1' matrix:

$$W^{(1)} = \begin{pmatrix} & W^{(0)} & \\ \frac{\partial}{\partial q_1}\varphi^{(0,1)} & \cdots & \frac{\partial}{\partial q_n}\varphi^{(0,1)} \\ \vdots & \vdots & \vdots \\ \frac{\partial}{\partial q_1}\varphi^{(0,\bar{N}_0)} & \cdots & \frac{\partial}{\partial q_n}\varphi^{(0,\bar{N}_0)} \end{pmatrix}_{(n+\bar{N}_0)\,x\,n}$$

$$\vec{K}^{(1)} = \begin{pmatrix} \vec{K}^{(0)} \\ \dot{q}_j\frac{\partial}{\partial q_j}\varphi^{(0,1)} \\ \vdots \\ \dot{q}_j\frac{\partial}{\partial q_j}\varphi^{(0,\bar{N}_0)} \end{pmatrix} \tag{20}$$

The constraints (17) hold for all times, then.

$\vec{E}^{(1)} = \vec{0}$ if $\vec{E}^{(0)} = \vec{0}$.

We next investigate for left zero modes of $W^{(1)}$, that is, eigenvectors $\vec{\omega}^{(1,k)}$ of its transpose with null eigenvalue, but now Nullity $W^{(1)T} = N_1 = n + \bar{N}_0 - p_1$ with $p_1 =$ rank $W^{(1)}$, $1 \leq p_1 \leq n$:

$$\vec{\omega}^{(1,r)}.W^{(1)} = 0, \quad r = 1,...,N_1 \tag{21}$$

The proper vectors $\vec{\omega}^{(1,k)}$ include those of the previous level, augmented by an appropriate number of zeroes [10], which are denoted by $\vec{\omega}^{(1,N_1-N_0+1)}, \vec{\omega}^{(1,N_1-N_0+2)},...,\vec{\omega}(1,N_1)$, they just reproduce the previous constraints and are therefore not considered. The remaining $N_1 - N_0 = \bar{N} + p_0 - p_1 = n - p_1 - d_0$ zero modes (if they exist), when contracted with $\vec{E}^{(1)}$ lead to expressions at 'level 1':

$$\Phi^{(1,k)} = \vec{\omega}^{(1,k)}.\vec{E}^{(1)} = \vec{\omega}^{(1,k)}.\vec{K}^{(1)}, \quad k = 1,...,N_1 - N_0 \tag{22}$$

with the first generation constraints:

$$\Phi^{(1,k)}(q,\dot{q}) = 0, \tag{23}$$

on the subspace of physical trajectories $(\vec{E}^{(0)} = \vec{0})$ However, not all the quantities (22) may be linearly independent [the functions (16) also must be considered], then one can obtain d_1 new gauge identities at level 1 of the form:

$$G^{(1,m_1)} = \vec{v}^{(1,m_1)}.\vec{E}^{(1)} - M^{m_1}_{\bar{m}_0}\varphi(0,\bar{m}_0) \equiv 0, \quad \bar{m}_0 = 1,...,\bar{N}_0,$$
$$m_1 = 1,...,d_1 \tag{24}$$

with:

$$\vec{v}(1,m_1) \equiv c_r(m_1)\vec{\omega}^{(1,r)}, \quad r = 1,...,N_1 - N_0 \tag{25}$$

In the subspace generated by $\vec{\omega}^{(1,k)}, k = 1,...,N_1 - N_0$ we have the vectors (25), then into it we construct $\bar{N}_1 = N_1 - N_0 - d_1$ vectors $\vec{u}^{(1,\bar{m}_1)}(q,\dot{q})$ to complete a base and we introduce the quantities:

$$\varphi(1,\bar{m}_1) = \vec{u}(1,\bar{m}_1).\vec{E}^{(1)} = \vec{u}(1,\bar{m}_1).\vec{K}^{(1)}, \quad \bar{m}_1 = 1,...,\bar{N}_1 \tag{26}$$

which represent, if $\vec{E}^{(0)} = \vec{0}$, genuine new constraints at level 1:

$$\varphi^{(1,\bar{m}_1)}(q,\dot{q}) = 0. \tag{27}$$

We now adjoin the new gauge identities (24) to the previous identities (15). With the functions (26) we proceed as before, adjoining their time derivative to (19) and construct $W^{(2)}$ as well as $\vec{K}^{(2)}$. This iterative process will terminate when there are no further zero modes [dim (Kernel $W^{(j)} = 0$ for some value of j] or if the constraints generated are linear combinations of the previous ones and hence lead to gauge identities only. At this point the algorithm has unraveled all the constraints of the Euler-Lagrange equations of motion.

We shall apply this matrix method to Lagrangians studied in [2, 4-6] to determine the corresponding local symmetry transformations for the case (4) when t is an ignorable variable. Thus we will see that each element of the maximal set of linearly independent gauge identities can be multiplied by an arbitrary function and the sum of these expressions permits to obtain the point symmetries, hence the number of independent arbitrary functions equals the number of gauge identities generated by the algorithm. This procedure can be carried out completely without the need of developing the Dirac-Hamilton formalism [2, 11, 12].

- Rothe [2]:

$$L = \frac{1}{2}\dot{q}_1^2 + \dot{q}_1 q_2 + \frac{1}{2}(q_1 - q_2)^2, \; n = 2, \tag{28}$$

$\vec{E}^{(0)}$ adopts the form (8) with:

$$W^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; p_0 = 1, \; \vec{K}^{(0)} = \begin{pmatrix} \ddot{q}_2 + q_2 - q_1 \\ -q_2 - \dot{q}_1 + q_1 \end{pmatrix},$$
$$N_0 = n - p_0 = 1, \tag{29}$$

and we have one constraint on Lagrangian level 0, then from (11, 12):

$$\vec{\omega}^{(0,1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \Phi^{(0,1)} = E_2^{(0)}, d_0 = 0 \quad \therefore \quad \vec{u}^{(0,1)} = \vec{\omega}^{(0,1)}, \tag{30}$$

without gauge identities:

$$\varphi^{(0,1)} = \Phi^{(0,1)} = E_2(0) = -q_2 - \dot{q}_1 + q_1, \; \bar{N}_0 = 1, \tag{31}$$

which gives us the genuine constraint $\varphi^{(0,1)} = 0$ on the subspace of physical paths. At the level 1, from (19, 20):

$$W^{(1)} = \begin{pmatrix} W^{(0)} \\ -1 \quad 0 \end{pmatrix}, \quad p_1 = 1, \quad \vec{K}^{(1)} = \begin{pmatrix} \vec{K}^{(0)} \\ \dot{q}_1 - \dot{q}_2 \end{pmatrix}, \quad N_1 = 2, \tag{32}$$

and (21) implies two zero modes:

$$\vec{\omega}^{(1,1)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{\omega}^{(1,2)} = \begin{pmatrix} \vec{\omega}^{(0,1)} \\ 0 \end{pmatrix}, \quad \Phi^{(1,2)} = \begin{pmatrix} \vec{\omega}^{(0,1)} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d}{dt}\varphi^{(0,1)} \end{pmatrix} = \varphi^{(0,1)}, \tag{33}$$

then the remaining quantity is:

$$\Phi^{(1,1)} = \vec{\omega}^{(1,1)} . \vec{E}^{(1)} = E_1^{(1)} + E_3^{(1)} = E_1^{(0)} + \frac{d}{dt}\varphi^{(0,1)} = E_1^{(0)} + \frac{d}{dt}E_2^{(0)} = q_2 + \dot{q}_1 - q_1, \tag{34}$$

but it is not independent because $\Phi^{(1,1)} = -\varphi^{(0,1)}$ that is:

$$E_1^{(0)} + E_2^{(0)} + \frac{d}{dt}E_2^{(0)} = 0, \quad d_1 = 1, \quad \bar{N}_1 = 0, \tag{35}$$

thus the matrix process terminates here at level 1 because it leads to gauge identities only.

We multiply (35) by $\varepsilon\, \alpha_1\, (t)$ to obtain:

$$\varepsilon\ \alpha_1 E_1^{(0)} + \varepsilon(\alpha_1 - \dot{\alpha}_1)E_2^{(0)} = -\frac{d}{dt}(\varepsilon\ \alpha_1 E_2^{(0)}), \tag{36}$$

then the comparison with (7) gives $\delta q_1 = \varepsilon\, \alpha_1$ and $\delta q_2 = \varepsilon(\alpha_1 - \dot{\alpha}_1)$, in harmony with the local symmetry transformations (2.5) in [2] for the case $\alpha_0 = 0$ with $\alpha_1\, (t)$ an arbitrary function.

- Henneaux-Teitelboim-Zanelli [2, 5]:

$$L = \frac{1}{2}(\dot{q}_2 - e^{q_1})^2 + \frac{1}{2}(\dot{q}_3 - q_2)^2, \quad n = 3, \tag{37}$$

$\vec{E}^{(0)}$ has the structure (8) such that:

$$W^{(0)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad p_0 = 2, \qquad \vec{K}^{(0)} = \begin{pmatrix} e^{q_1}(\dot{q}_2 - e^{q_1}) \\ -e^{q_1}\dot{q}_1 + \dot{q}_3 - q_2 \\ -\dot{q}_2 \end{pmatrix}, \qquad N_0 = 1, \tag{38}$$

with one constraint at level 0, without gauge identities ($(d_0 = 0, \bar{N}_0 = 1)$:

$$\vec{\omega}^{(0,1)} = \vec{u}^{(0,1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \Phi^{(0,1)} = \varphi^{(0,1)} = E_1{}^{(0)} = e^{q_1}(\dot{q}_2 - e^{q_1}). \tag{39}$$

At the level 1:

$$W^{(1)} = \begin{pmatrix} & W^{(0)} & \\ 0 & e^{q_1} & 0 \end{pmatrix}, \qquad p_1 = 2, \qquad \vec{K}^{(1)} = \begin{pmatrix} \vec{K}^{(0)} \\ \dot{q}_1 - \dot{q}_2 \end{pmatrix}, \qquad N_1 = 2,$$
$$\vec{\omega}^{(1,1)} = \begin{pmatrix} 0 \\ -e^{q_1} \\ 0 \\ 1 \end{pmatrix}, \qquad \vec{\omega}^{(1,2)} = \begin{pmatrix} \vec{\omega}^{(0,1)} \\ 0 \end{pmatrix}, \qquad \Phi^{(1,2)} = \varphi^{(0,1)}, \tag{40}$$

And

$$\Phi^{(1,1)} = -e^{q_1} E_2{}^{(1)} + E_4{}^{(1)} = -e^{q_1} E_2{}^{(0)} + \frac{d}{dt}\varphi^{(0,1)} = -e^{q_1} E_2{}^{(0)} + \frac{d}{dt} E_1{}^{(0)},$$
$$= e^{q_1}(-e^{q_1}\dot{q}_1 - \dot{q}_3 + q_2 + \dot{q}_1\dot{q}_2) = \dot{q}_1\varphi^{(0,1)} + e^{q_1}(q_2 - \dot{q}_3),$$

where the first term is proportional to (39), then we can continue the process with the second term:

$$\varphi^{(1,1)} = e^{q_1}(q_2 - \dot{q}_3) = \Phi^{(1,1)} - \dot{q}_1\varphi^{(0,1)} = -\dot{q}_1 E_1{}^{(0)} - e^{q_1} E_2{}^{(0)} + \frac{d}{dt} E_1{}^{(0)}. \tag{41}$$

At the level 2:

$$W^{(2)} = \begin{pmatrix} & W^{(1)} & \\ 0 & 0 & -e^{q_1} \end{pmatrix}, \qquad p_2 = 2, \qquad \vec{K}^{(2)} = \begin{pmatrix} \vec{K}^{(1)} \\ e^{q_1}(\dot{q}_1 q_2 + \dot{q}_2 - \dot{q}_1\dot{q}_3) \end{pmatrix},$$
$$\vec{\omega}^{(2,1)} = \begin{pmatrix} 0 \\ 0 \\ e^{q_1} \\ 0 \\ 1 \end{pmatrix}, \qquad \vec{\omega}^{(2,2)} = \begin{pmatrix} \vec{\omega}^{(1,1)} \\ 0 \end{pmatrix}, \qquad \vec{\omega}^{(2,3)} = \begin{pmatrix} \vec{\omega}^{(1,2)} \\ 0 \end{pmatrix}, \tag{42}$$

$$\vec{\omega}^{(2,1)} \cdot \vec{E}^{(2)} = e^{q_1} E_3{}^{(2)} + E_5{}^{(2)} = e^{q_1} E_3{}^{(1)} + \frac{d}{dt}\varphi^{(1,1)} = \dot{q}_1 e^{q_1}(q_2 - \dot{q}_3) = \dot{q}_1\varphi^{(1,1)},$$

which implies the gauge identity:

$$\dot{q}_1^2 E_1{}^{(0)} + \dot{q}_1 e^{q_1} E_2{}^{(0)} + e^{q_1} E_3{}^{(0)} - \dot{q}_1 \frac{d}{dt} E_1{}^{(0)} + \frac{d}{dt}\left(-\dot{q}_1 E_1{}^{(0)} - e^{q_1} E_2{}^{(0)} + \frac{d}{dt} E_1{}^{(0)}\right) = 0. \tag{43}$$

We multiply (43) by $\varepsilon\, \alpha_3\,(t)\, e^{-q1}$ to deduce the expression:

$$\varepsilon\, \ddot{\alpha}_3 e^{-q_1} E_1{}^{(0)} + \varepsilon\, \dot{\alpha}_3 E_2{}^{(0)} + \varepsilon\, \alpha_3 E_3{}^{(0)} = \varepsilon\, \frac{d}{dt}[\alpha_3(\dot{q}_3 - q_2) + \dot{\alpha}_3(\dot{q}_2 - e^{q_1})], \tag{44}$$

whose comparison with (7) permits to obtain the point symmetry transformations (2.7) in [2] $\delta q_1 = \varepsilon \ddot{\alpha}_3 e^{-q1}, \delta q_2 = \varepsilon\, \dot{\alpha}_3$ and $\delta q_3 = \varepsilon\, \alpha_3$ for $\alpha_0 = 0$ and $\alpha_3\,(t)$ an arbitrary function.

- Havelková [4]-Torres del Castillo [6]:

$$L = (\dot{q}_1 - q_2)\dot{q}_3 + q_1\, q_3, \quad n = 3 \tag{45}$$

In this case:

$$W^{(0)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad p_0 = 2, \qquad \vec{K}^{(0)} = \begin{pmatrix} -q_3 \\ \dot{q}_3 \\ -\dot{q}_2 - q_1 \end{pmatrix}, \qquad N_0 = 1,$$

$$\vec{\omega}^{(0,1)} = \vec{u}^{(0,1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \Phi^{(0,1)} = \varphi^{(0,1)} = E_2{}^{(0)} = \dot{q}_3, \quad W^{(1)} = \begin{pmatrix} & W^{(0)} & \\ 0 & 0 & 1 \end{pmatrix}, \quad p_1 = 2,$$

$$\vec{K}^{(1)} = \begin{pmatrix} \vec{K}^{(0)} \\ 0 \end{pmatrix}, \qquad \vec{\omega}^{(1,1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \qquad \vec{\omega}^{(1,2)} = \begin{pmatrix} \vec{\omega}^{(0,1)} \\ 0 \end{pmatrix}, \qquad \Phi^{(1,2)} = \varphi^{(0,1)}, \tag{46}$$

then

$$\Phi^{(1,1)} = E_1{}^{(1)} - E_4{}^{(1)} = E_1{}^{(0)} - \frac{d}{dt}E_2{}^{(0)} = -q_3 = \varphi^{(1,1)}, \tag{47}$$

and it is necessary to go towards level 2:

$$W^{(2)} = \begin{pmatrix} & W^{(1)} & \\ 0 & 0 & 0 \end{pmatrix}, \ p_2 = 2, \ \vec{K}^{(2)} = \begin{pmatrix} \vec{K}^{(1)} \\ -\dot{q}_3 \end{pmatrix}, \quad \vec{\omega}^{(2,1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{\omega}^{(2,2)} = \begin{pmatrix} \vec{\omega}^{(1,1)} \\ 0 \end{pmatrix}, \quad \vec{\omega}^{(2,3)} = \begin{pmatrix} \vec{\omega}^{(1,2)} \\ 0 \end{pmatrix}, \tag{48}$$

such that:

$$\Phi^{(2,1)} = E_5{}^{(2)} = \frac{d}{dt}\varphi^{(1,1)} = \frac{d}{dt}\left(E_1{}^{(0)} - \frac{d}{dt}E_2{}^{(0)}\right) = -\dot{q}_3 = -\varphi^{(0,1)} = -E_2{}^{(0)},$$

thus we have the gauge identity:

$$E_2^{(0)} + \frac{d}{dt}\left(E_1^{(0)} - \frac{d}{dt}E_2^{(0)}\right) = 0, \tag{49}$$

We multiply (49) by $\varepsilon\,\alpha$ to obtain the relation:

$$\varepsilon\,\dot{\alpha}E_1{}^{(0)} + \varepsilon\,(\ddot{\alpha} - \alpha)\,E_2{}^{(0)} = \varepsilon\,\frac{d}{dt}\left[\dot{\alpha}\,E_2{}^{(0)} + \alpha\left(E_1{}^{(0)} - \frac{d}{dt}E_2{}^{(0)}\right)\right],$$

then from (7):

$$\delta q_1 = \varepsilon\,\dot{\alpha}, \quad \delta q_2 = \varepsilon(\ddot{\alpha} - \alpha), \quad \delta q_3 = 0, \tag{50}$$

and, for example, if $\alpha = c_2\,e^t - c_3\,e^{-t}$ we deduce the expressions of [4] p. 28 and (30) in [6] for the particular case $c_1 = f = 0$.

- Rothe [2]:

$$L = \frac{1}{2}\dot{q}_1^2 + (q_2 - q_3)\dot{q}_1 + \frac{1}{2}(q_1 - q_2 + q_3)^2, \ \ n = 3. \tag{51}$$

At the level 0:

$$W^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad p_0 = 1, \quad \vec{K}^{(0)} = \begin{pmatrix} \dot{q}_2 - \dot{q}_3 - q_1 + q_2 - q_3 \\ -\dot{q}_1 + q_1 - q_2 + q_3 \\ \dot{q}_1 - q_1 + q_2 - q_3 \end{pmatrix}, \quad \vec{\omega}^{(0,1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{\omega}^{(0,2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\Phi^{(0,1)} = E_2{}^{(0)} = -\dot{q}_1 + q_1 - q_2 + q_3, \qquad \Phi^{(0,2)} = E_3{}^{(0)} = \dot{q}_1 - q_1 + q_2 - q_3 = -\Phi^{(0,1)},$$

with the gauge identity $\Phi^{(0,1)} + \Phi^{(0,2)} = 0,$ that is:

$$G^{(0,1)} = \left(\vec{\omega}^{(0,1)} + \vec{\omega}^{(0,2)}\right) \cdot \vec{E}^{(0)} = \vec{v}^{(0,1)} \cdot \vec{E}^{(0)} = E_2{}^{(0)} + E_3{}^{(0)} = 0, \qquad \vec{v}^{(0,1)} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \tag{52}$$

and we may employ the vector:

$$\vec{u}^{(0,1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \therefore \quad \varphi^{(0,1)} = E_2^{(0)}.$$

At level 1:

$$W^{(1)} = \begin{pmatrix} & W^{(0)} & \\ -1 & 0 & 0 \end{pmatrix}, \; p_1 = 1, \vec{K}^{(1)} = \begin{pmatrix} \vec{K}^{(0)} \\ \dot{q}_1 - \dot{q}_2 + \dot{q}_3 \end{pmatrix}, \vec{\omega}^{(1,1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \vec{\omega}^{(1,2)} = \begin{pmatrix} \vec{\omega}^{(0,1)} \\ 0 \end{pmatrix}, \vec{\omega}^{(1,3)} = \begin{pmatrix} \vec{\omega}^{(0,2)} \\ 0 \end{pmatrix},$$

$$\Phi^{(1,1)} = E_1{}^{(0)} + \frac{d}{dt}\varphi^{(0,1)} = E_1{}^{(0)} + \frac{d}{dt}E_2{}^{(0)} = \dot{q}_1 - q_1 + q_2 - q_3 = -\varphi^{(0,1)} = -E_2{}^{(0)},$$

with the gauge identity:

$$E_1^{(0)} + E_2^{(0)} + \frac{d}{dt}E_2^{(0)} = 0. \tag{53}$$

We multiply (52) and (53) by $\varepsilon\,\alpha$ and $\varepsilon\,\alpha_1$, respectively and we add the corresponding expressions to obtain:

$$\varepsilon\,\alpha_1\,E_1{}^{(0)} + \varepsilon\,(\alpha_3 - \dot{\alpha}_1 + \alpha_1)\,E_2{}^{(0)} + \varepsilon\,\alpha_3\,E_3{}^{(0)} = -\frac{d}{dt}\left(\varepsilon\,\alpha_1\,E_2{}^{(0)}\right), \tag{54}$$

and thus from (7) we reproduce the local symmetry transformations (2.36) in [2] because $\delta q_1 = \varepsilon\,\alpha_1,\ \delta q_2 = \varepsilon\,(\alpha_1 - \dot{\alpha}_1 + \alpha_3)$ and $\delta q_3 = \varepsilon\,\alpha_3$.

The transformations (36, 44, 50, 54) show the existence of one gauge identity by each arbitrary function present into the point symmetry and also the presence of genuine constraints.

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