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Some Applications of Complete Bell Polynomials

[1]A. Zúñiga-Segundo, [2]J. López-Bonilla and [2]S. Vidal-Beltrán

[1]ESFM, Instituto Politécnico Nacional, Edif. 9, Dpto. Física,
Lindavista 07738, CDMX, México,
[2]ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4,
1er. Piso, Col. Lindavista CP 07738 CDMX, México

Abstract: We exhibit that the coefficients of the characteristic equation of any matrix A_{nxn}, the Stirling numbers of the first kind and the quantities $\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{k^m}$ can be written in terms of the complete Bell polynomials.

Key words: Bell polynomials · Stirling numbers · Characteristic equation · Harmonic sums

INTRODUCTION

In [1, 2] we find the following expression for the Bell polynomials:

$$Y_m(x_1,x_2,\dots,x_m) = \begin{vmatrix} \binom{m-1}{0}x_1 & \binom{m-1}{1}x_2 & \cdots & \binom{m-1}{m-2}x_{m-1} & \binom{m-1}{m-1}x_m \\ -1 & \binom{m-2}{0}x_1 & \cdots & \binom{m-2}{m-3}x_{m-2} & \binom{m-2}{m-2}x_{m-1} \\ 0 & -1 & \cdots & \binom{m-3}{m-4}x_{m-3} & \binom{m-3}{m-3}x_{m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \binom{1}{0}x_1 & \binom{1}{1}x_2 \\ 0 & 0 & \cdots & -1 & \binom{0}{0}x_1 \end{vmatrix} \tag{1}$$

therefore:

$$Y_0 = 1,\ \ Y_1 = x_1,\ \ Y_2 = x_1^2 + x_2,\ \ Y_3 = x_1^3 + 3x_1x_2 + x_3\,,\ \ Y_4 = x_1^4 + 6\,x_1^2\,x_2 + 4\,x_1x_3 + 3\,x_2^2 + x_4, \tag{2}$$

$$Y_5 = x_1^5 + 10\,x_1^3\,x_2 + 10\,x_1^2\,x_3 + 15\,x_1\,x_2^2 + 5\,x_1\,x_4 + 10\,x_2x_3 + x_5,$$

In Sec. 2 we exhibit that (1) allows generate the coefficients of the characteristic equation of any square matrix. In Sec. 3 we indicate that the Stirling numbers of the first kind $S_n^{(m)}$ [3-5] and the quantities [6]:

$$S_n(m) \equiv \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{k^m}, \qquad m,n \geq 1, \tag{3}$$

can be written in terms of (1).

Characteristic Equation: For an arbitrary matrix $\mathbf{A}_{nxn} = (A^i{}_j)$ its characteristic equation [7-9]:

Corresponding Autor: J. López-Bonilla, ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista CP 07738 CDMX, México. E-mail: jlopezb@ipn.mx.

$$\lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0, \tag{4}$$

can be obtained, through several procedures [7, 10-13], directly from the condition det $(A^i{}_j - \lambda \delta^i_j) = 0.$ The approach of Leverrier-Takeno [10, 14-19] is a simple and interesting technique to construct (4) based in the traces of the powers $\mathbf{A}^r,\ r = 1, \ldots, n.$ In fact, if we define the quantities:

$$a_0 = 1, \qquad s_k = tr\, \mathbf{A}^k, \qquad k = 1, 2, \ldots, n, \tag{5}$$

then the process of Leverrier-Takeno implies (4) wherein the α_i are determined with the recurrence relation:

$$r\, a_r + s_1\, a_{r-1} + s_2\, a_{r-2} + \ \ldots + s_{r-1}\, a_1 + s_r = 0, \qquad r = 1, 2, \ldots, n, \tag{6}$$

therefore:

$$\begin{gathered} a_1 = -s_1\,, \quad 2!\ a_2 = (s_1)^2 - s_2\,, \quad 3!\ a_3 = -(s_1)^3 + 3\, s_1\, s_2 - 2\, s_3\,, \\ 4!\ a_4 = (s_1)^4 - 6\,(s_1)^2\, s_2 + 8\, s_1\, s_3 + 3\,(s_2)^2 - 6\, s_4\,, \\ 5!\ a_5 = -(s_1)^5 + 10\,(s_1)^3\, s_2 - 20\,(s_1)^2\, s_3 - 15\, s_1 (s_2)^2 + 30\, s_1\, s_4 + 20\, s_2\, s_3 - 24\, s_5,\ \ldots \end{gathered} \tag{7}$$

in particular, det $\mathbf{A} = (-1)^n\, a_n$, that is, the determinant of any matrix only depends on the traces s_r, which means that $\mathbf{A}$ and its transpose have the same determinant.

In [20-22] we find the general expression:

$$a_m = \frac{(-1)^m}{m!} \begin{vmatrix} s_1 & s_2 & s_3 & \cdots & s_{m-1} & s_m \\ m-1 & s_1 & s_2 & \cdots & s_{m-2} & s_{m-1} \\ 0 & m-2 & s_1 & \cdots & s_{m-3} & s_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & s_1 \end{vmatrix}, \qquad m = 1, \ldots, n\,. \tag{8}$$

which allows reproduce the expressions (7). Now we can see that the formula (8) permits relate the coefficients of the characteristic equation (4) with the complete Bell polynomials [3, 5, 23-25]; with (2) we can deduce (7) if we employ $x_1 = -s_1,\quad x_2 = -s_2,\quad x_3 = -2\, s_3,\quad x_4 = -6\, s_4\,,\ x_5 = -24\, s_5, \ldots$, that is [25, 26]:

$$a_m = \frac{1}{m!}\, Y_m(-0!\ s_1, -1!\ s_2, -2!\, s_3, -3!\ s_4, \ldots, -(m-2)!\ s_{m-1}, -(m-1)!\ s_m). \tag{9}$$

In fact, it is simple to prove that (1) with $x_k = -(k-1)!\ S_k$ implies (8), thus the coefficients of the characteristic equation (4) are generated by the complete Bell polynomials.

Stirling Numbers of the First Kind: Batir [6] studies (3), then here we show that his results in terms of harmonic sums can be written with the help of Bell polynomials. In fact, for example, he obtains the expressions:

$$S_n(1) = H_n\,, \qquad S_n(2) = \tfrac{1}{2}\left(H_n^2 + H_n^{(2)}\right), \qquad S_n(3) = \tfrac{1}{6}\left(H_n^3 + 3\, H_n\, H_n^{(2)} + 2\, H_n^{(3)}\right), \tag{10}$$

$$S_n(4) = \frac{1}{24}\left[H_n^4 + 6\, H_n^2\, H_n^{(2)} + 8\, H_n\, H_n^{(3)} + 3\,(H_n^{(2)})^2 + 6\, H_n^{(4)}\right], \tag{11}$$

$$S_n(5) = \frac{1}{120}\Big[H_n^5 + 10\, H_n^3\, H_n^{(2)} + 20\, H_n^2\, H_n^{(3)} + 15\, H_n\, (H_N^{(2)})^2 + 30\, H_n\, H_n^{(4)} + 20\, H_n^{(2)} H_n^{(3)} + 24\, H_n^{(5)}\Big],$$

with the participation of the harmonic sums [4, 27-29]:

$$H_n^{(k)} \equiv \sum_{r=1}^{n} \frac{1}{r^k}\,, \qquad k \geq 1, \qquad H_n^{(1)} = H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\,. \tag{12}$$

The relations (10) were also deduced by Flajolet-Sedgewick [30].

The expressions (2) allow reproduce the relations (10) and (11) if we employ $x_r = (r-1)!\, H_n^{(r)}$, $r = 1, \ldots m$, thus [2, 25, 31]:

$$S_n(m) = \frac{1}{m!}\, Y_m\Big(H_n, 1!\, H_n^{(2)}, 2!\, H_n^{(3)}, \ldots, (m-1)!\, H_n^{(m)}\Big). \tag{13}$$

On the other hand, we know that the Bell polynomials give the Stirling numbers of the first kind [3-5] via the formula [23, 31, 32]:

$$S_{n+1}^{(m+1)} = \frac{(-1)^{m+n}\, n!}{m!}\, Y_m\Big(H_n, -1!\, H_n^{(2)}, 2!\, H_n^{(3)}, \ldots, (m-1)!\, H_n^{(m)}\Big), \tag{14}$$

hence (13) and (14) are unified in the following relation:

$$Y_m\Big(H_n, \lambda\, H_n^{(2)}, 2!\, H_n^{(3)}, \ldots, (m-1)!\, H_n^{(m)}\Big) = \begin{cases} m!\; S_n(m), & \lambda = 1, \\ \frac{(-1)^{m+n}\, m!}{n!}\, S_{n+1}^{(m+1)}, & \lambda = -1, \end{cases} \tag{15}$$

which shows to the Bell polynomials as generators of the quantities (3) and the Stirling numbers.

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