World Engineering & Applied Sciences Journal 9 (3): 86-88, 2018
ISSN 2079-2204

DOI: 10.5829/idosi.weasj.2018.86.88

Integration of the Descending Factorial Function

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Abstract: We study the quantities $\int_0^{-1}\binom{x}{k}\,dx$ which allow determine $\sum_{r=1}^{n}\frac{(-1)^r}{r}S_n^{(r)}$ for the Stirling numbers of the first kind.

Key words: Bernoulli and Stirling numbers · Characteristic polynomial · Binomial coefficient

INTRODUCTION

Here we have interest in the quantities:

$$A_n = \sum_{r=1}^{n}\frac{(-1)^r}{r}S_n^{(r)}, \qquad n \geq 1, \tag{1}$$

for the Stirling numbers of the first kind [1-4]. In Sec. 2 we deduce a recurrence relation for (1) whose solution involves the integrals:

$$C_k = \int_0^{-1}\binom{x}{k}\,dx, \qquad k \geq 1, \tag{2}$$

which are obtained in terms of the Bernoulli numbers [1, 5-8].

Descending Factorial Function: The descending factorial function $[x]_n$ is a polynomial of degree n in x, given by [1]:

$$[x]_n = n!\binom{x}{n}x\,(x-1)\,(x-2)\cdots(x-(n-2))\,(x-(n-1)) = \sum_{m=1}^{n} S_n^{(m)}x^m, \qquad n \geq 1, \tag{3}$$

thus:

$$\int_0^{\xi}[x]_n\,dx = \sum_{m=1}^{n}\frac{\xi^{m+1}}{m+1}S_n^{(m)} = \sum_{j=2}^{n+1}\frac{\xi^j}{j}S_n^{(j-1)} = \sum_{j=2}^{n+1}\frac{\xi^j}{j}\left(n\,S_n^{(j)} + S_{n+1}^{(j)}\right),$$

$$= n\left(\sum_{j=1}^{n}\frac{\xi^j}{j}S_n^{(j)} - \xi\,S_n^{(1)}\right) + \sum_{j=1}^{n+1}\frac{\xi^j}{j}S_{n+1}^{(j)} - \xi\,S_{n+1}^{(1)}, \qquad S_n^{(n+1)} = 0, \qquad S_{n+1}^{(1)} = (-1)^n\,n!$$

therefore:

$$\sum_{j=1}^{n+1}\frac{\xi^j}{j}S_{n+1}^{(j)} + n\sum_{r=1}^{n}\frac{\xi^j}{j}S_n^{(j)} = \int_0^{\xi}[x]_n\,dx, \qquad n \geq 1, \tag{4}$$

where we can employ $\xi = -1$ to obtain the following recurrence relation for (1):

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$$A_{n+1} + n\,A_n = \int_0^{-1} [x]_n\ dx, \qquad n = 1, 2, ... \tag{5}$$

From (1) we have that $A_1 = -\,S_1^{(1)} = -1$, then the solution of (5) is given by:

$$A_n = (-1)^n\ (n-1)!\left[1 - \sum_{k=1}^{n-1}\ (-1)^k\ \int_0^{-1}\binom{x}{k}\ dx\ \right], \qquad n \geq 2, \tag{6}$$

with the participation of (2), that is, $C_1 = \frac{1}{2},\ C_2 = -\frac{5}{12}\ ,\ C_3 = \frac{3}{8}\ , ...,$ hence $A_2 = \frac{3}{2}\ ,\ A_3 = -\frac{23}{6}\ ,$ etc.

The expression $k!\binom{x}{k}$ is the characteristic polynomial of the matrix $\mathbf{A} = Diag\ (0, 1, 2, ..., k-1)$, whose coefficients can be constructed via the Leverrier-Takeno's method [9, 10]:

$$k!\ \binom{x}{k} = x^k +\ a_1\ x^{k-1} + a_2\ x^{k-2} + \cdots + a_{k-1}\ x, \qquad a_k = (-1)^k \det \mathbf{A} = 0, \tag{7}$$

in fact, we define the quantities:

$$a_0 = 1, \qquad s_q = tr\ \mathbf{A}^q, \qquad q = 1, 2, ..., k \tag{8}$$

then the α_i are determined with the Newton's recurrence relation:

$$r\ a_r + s_1\ a_{r-1} + s_2\ a_{r-2} +\ ...\ + s_{r-1}\ a_1 + s_r = 0, \qquad r = 1, 2, ..., k, \tag{9}$$

therefore:

$$a_1 = -s_1\ , \qquad 2!\ a_2 = (s_1)^2 - s_2\ , \qquad 3!\ a_3 = -(s_1)^3 + 3\ s_1\ s_2 - 2\ s_3\ ,$$
$$4!\ a_4 =\ (s_1)^4 - 6\ (s_1)^2\ s_2 + 8\ s_1\ s_3 + 3\ (s_2)^2 - 6\ s_4\ , \quad etc. \tag{10}$$

In [11, 12] we find the general expression:

$$a_r = \frac{(-1)^r}{r!}\begin{vmatrix} s_1 & r-1 & 0 & \cdots & 0 \\ s_2 & s_1 & r-2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{r-1} & s_{r-2} & \cdots & \cdots & 1 \\ s_r & s_{r-1} & \cdots & \cdots & s_1 \end{vmatrix}, \qquad r = 1, ..., k\ . \tag{11}$$

In this case the traces s_q are sums of powers of integers that can be written in terms of the Bernoulli numbers [1, 5-8]:

$$s_q = \sum_{j=1}^{k-1}\ j^q = \frac{k^{q+1}}{q+1}\ \sum_{l=0}^{q}\ \binom{q+1}{l}\ \frac{B_l}{k^l}\ , \qquad B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}\ , ... \tag{12}$$

thus (7) allows obtain the quantities (2):

$$\int_0^{-1}\binom{x}{k}\ dx = \frac{(-1)^{k+1}}{k!}\ \sum_{r=0}^{k-1}\ \frac{(-1)^r}{k+1-r}\ a_r\ , \qquad k \geq 1, \tag{13}$$

of interest in (6).

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