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Fatigue Failure of an Oval Cross Section Prismatic Bar at Pulsating Torsion

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Abstract: The numbers of pulsating torsions, at which the first damages and fatigue failure of an oval cross section prismatic bar happen, are determined. The bar's material has no strengthening and the plastic area entirely covers the section's contour. At the initial elastico-plastic torsion of the bar from the natural state, V.V.Sokolovski's solution was used. Intensity of residual deformations is accepted for a determining fatigue failure parameter.

Key words: Oval section bar · Pulsating torsion · Damage · Fatigue failure

INTRODUCTION

Here, under fatigue or cyclic durability it is commonly understood discontinuities of the material of construction under consideration under cyclic change of plastic deformation [1-3]. At some experimental investigations [4] carried out by infrared spectroscopy and acoustic emission it is noted the fact that first fatigue damages of structural elements appear after certain number of cycles after the start of cyclic deformation process. The number of cycles of loading, preceding to formation of first damages and to failure are commensurable values. Here, by using the start of cyclic damage conditions and cyclic strength obtained in [5], a problem of fatigue failure of an oval cross section prismatic bar at pulsating torsion is solved. Herewith, the strengthening of the bar material is excluded (ideally elastico- plastic deformable material is considered) and it is assumed that the plastic area entirely covers the bar contour the known solution of V.V.Sokolovskii [1] is used. In the unloading process appearance of the area of secondary plastic deformations is allowed [3].

The residual stress and strains are found by using the V.V. Moskvitin theorems on secondary plastic deformations [3].

Statement and Solution of V.V. Sokolovsky Problem on Initial Torsion of a Bar: Let as consider an oval cross section (almost elliptic) prismatic bar whose contour is determined by the following equations in the parametric form [1].

$$x = (a+2b)\sin\varphi + b\sin\varphi\cos 2\varphi,$$
$$y - = (a-2b)\cos\varphi + b\cos\varphi\cos 2\varphi,$$

where φ is a slope tangential to the section contour, $a > 3b$ Herewith, the semi-axes of the oval (ellipse) will be: $a + b$, $a - b$. A quarter of the considered oval cross section is given in Fig. 1.

Let the considered bar in the natural state be subjected to the torque M'. Under the action of the torque M' in the bar there appear the stresses σ'_{xz}, σ'_{yz} strains ε_{xz}', ε_{yz}' and displacements u'_x, u'_y, u'_z. The coordinate system $x\ y\ z$, whose origin coincides with the center of the oval, the axis x with the axis of the bar, the axis z with great small semi- axis, the axis y with small semi-axis, is used. The problem is in determination of stress, strain and displacements fields in elastic and plastic domains and also in establishment of boundaries between these domains. The given problem, the problem of initial elastico - plastic torsion of a bar under suppositions that the bar material has no strengthening and the plastic area entirely covers the cross section, was solved by V.V. Sokolovsky by the specific, so called the inverse method.

It is assumed that the hypothesis of plane cross sections holds. Therewith the displacements u'_x, u'_y, u'_z may be represented in the form.

$$u'_* = -\theta' y\ z; \quad u'_y = \theta' x\ z;\ u'_z = \theta' f(x,y)\,, \tag{1}$$

where θ' is the twist angle of a unit length bar area, $f(x, z)$ is a torsion function characterizing the warping. Relative torsion angle ϑ' is considered to be positive for definiteness.

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Since the stresses $\sigma'_{xx} = \sigma'_{yy} = \sigma'_{zz} = \sigma'_{xy} = 0$ and strains $\sigma'_{xx} = \sigma'_{yy} = \sigma'_{zz} = \sigma'_{xy} = 0$, then the equilibrium equalities and conditions of strain compatibility accept the form:

$$\frac{\partial \sigma'_{xz}}{\partial x} + \frac{\partial \sigma'_{yz}}{\partial y} = 0, \quad (2)$$

$$\frac{\partial \varepsilon'_{yz}}{\partial x} + \frac{\partial \varepsilon'_{xz}}{\partial y} = \theta'. \quad (3)$$

The Cauchy kinematic relations should be fulfilled:

$$\varepsilon'_{xz} = \frac{1}{2}\left(\frac{\partial u'_x}{\partial y} + \frac{\partial u'_y}{\partial x}\right); \quad \varepsilon'_{yz} = \frac{1}{2}\left(\frac{\partial u'_y}{\partial z} + \frac{\partial u'_z}{\partial y}\right) \quad (4)$$

Since the lateral surface of the bar is stress free, on the cross section contour the tangential stress vector should be directed along the tangent to the contour:

$$\frac{\sigma'_{yz}}{\sigma'_{xz}} = \frac{dy}{dx} = tg\varphi$$

The principal momentum of tangential stresses acting on the bar cross section equals the torque M':

$$\iint_F \left(x\sigma'_{yz} - y\sigma'_{xz}\right) dF = M', \quad (5)$$

Here F is the area of bar cross section.

In the elastic domain, the Hook's law holds:

$$\sigma'_{xz} = 2G\varepsilon'_{xz} \quad \sigma'_{yz} = 2G\varepsilon'_{yz}, \quad (6)$$

where G is a shear modulus?

In the plastic domain, the von Mises yield criterion is fulfilled:

$$\left[\sigma'^2_{xz} + \sigma'^2_{yz}\right]^{1/2} = \tau_s, \quad (7)$$

where τ_s is the yield point in shear expressed by the yield point in tension σ_s by the relation? $\tau_s = \sigma_s / \sqrt{3}$.

Furthermore, the continuity condition of tangential stress and axial displacement components when passing through the boundary of elastic and plastic domains is fulfilled.

According to [1], the solution of the elastic problem has the form:

$$\theta' = \frac{2}{\pi G a^4}\left[1 - 5\frac{b^2}{a^2} + \frac{51}{8}\frac{b^4}{a^4}\right]^{-1} M' \quad (8)$$

or

$$M' = \left(1 - 5\frac{b^2}{a^2} + \frac{51}{8}\frac{b^4}{a^4}\right)\frac{\pi G a^4}{2}\theta', \quad (9)$$

$$f(x,y) = -2\frac{b}{a}xy, \quad (10)$$

$$\sigma'_{xz} = -\frac{G(a+2b)}{a}y\theta'; \quad \sigma'_{yz} = \frac{G(a-2b)}{a}x\theta', \quad (11)$$

$$\varepsilon'_{xz} = -\frac{a+2b}{2a}y\theta'; \quad \varepsilon'_{yz} = -\frac{a-2b}{2a}x\theta'. \quad (12)$$

The displacement u'_x, u'_y, u'_z are determined from formulas (1) taking into account (8) and (10).

In conformity to the considered problem, the intensity of the strain ε'_t is expressed by the formula:

$$\varepsilon'_t = \frac{2}{\sqrt{3}}\left(\varepsilon'^2_{xz} + \varepsilon'^2_{yz}\right)^{1/2}. \quad (13)$$

By using (12) in formula (13) the intensity of elastic deformations will be:

$$\varepsilon'_t = \frac{\theta'}{\sqrt{3}a}\left[(a+2b)^2 y^2 + (a-2b)^2 x^2\right]^{1/2}. \quad (14)$$

Let for $\theta = \theta_s$ on the boundary of the cross section contour first the plastic deformations appear. It is clear that the twist θ first will accept the value θ_s at the point $x = 0$, $y = \pm (a - b)$. At this point, the yield condition (7) has the form $|\sigma'_{xz}| = \tau_s$. Allowing for the first formula of (11) we get:

$$\theta_s = \frac{a\tau_s}{G(a-b)(a+2b)}. \quad (15)$$

The torque M_s at which plastic deformations appear, based on relations (9) and (15) are expressed by the formula:

$$M_s = \left(1 - 5\frac{b^2}{a^2} + \frac{51}{8}\frac{b^4}{a^4}\right)\frac{\pi a^5 \tau_s}{2(a-b)(a+2b)}. \quad (16)$$

For $M' > M_s$, in the cross section of the bar there arises a plastic deformation area. In the case when the plastic area entirely covers the cross section, it holds the V.V. Sokolovskii solution [1]. According to V.V.Sokolovskii's inverse method, the elastico plastic boundary is given in the form of an ellipse

$$\frac{x^2}{c^2} + \frac{y^2}{d'^2} = 1$$

Herewith, for the semi-axis c' and d' the following formulas are obtained:

$$\left.\begin{matrix} c \\ d' \end{matrix}\right\} = 2b\left[\frac{\tau_s}{4Gb\theta'} + \sqrt{1+\left(\frac{\tau_s}{4Gb\theta'}\right)^2} \pm 1\right], \quad (17)$$

Where the twist θ' and the torque M' are connected with the formula:

$$M' = \frac{1}{3}\pi\tau_s\left(2a^3 - 9ab^2 + 8b^3\right) - \frac{8}{3}\pi\tau_s b^3 \times$$
$$\times\left\{1 + 2\left(\frac{\tau_s}{4Gb\theta'}\right)^3 - \left[1 - 2\left(\frac{\tau_s}{4Gb\theta'}\right)^2\right]\left[1+\left(\frac{\tau_s}{4Gb\theta'}\right)^2\right]^{1/2}\right\} \quad (18)$$

In the elastic area (inside the ellipse) the displacement u' is represented by the formula:

$$u'_z = -\theta' x y\left[\left(1+\left(\frac{\tau_s}{4G\theta' b}\right)^2\right)^{1/2} - \frac{\tau_3}{4G\theta' b}\right], \quad (19)$$

In the plastic area by the formula:

$$u'_z = 8b^2\theta'\cos\varphi\left\{\frac{x}{2b} - \left[1+\left(1+\left(\frac{\tau_s}{4G\theta' b}\right)^2\right)^{1/2}\right]\sin\varphi\right\} \quad (20)$$

In the elastic area, the deformations ε'_{xz} and ε'_{yz} are expressed by the following formula:

$$\varepsilon'_{xz} = -\frac{\theta' y}{2}\left\{1 + \left[1+\left(\frac{\tau_s}{4Gb\theta'}\right)^2\right]^{1/2} - \frac{\tau_s}{4Gb\theta'}\right\}, \quad (21)$$

$$\varepsilon'_{yz} = -\frac{\theta' x}{2}\left\{1 - \left[1+\left(\frac{\tau_s}{4Gb\theta'}\right)^2\right]^{1/2} + \frac{\tau_s}{4Gb\theta'}\right\}, \quad (22)$$

In the plastic area we have

$$\varepsilon'_{yz} = \frac{\theta'}{2}x,$$
$$\varepsilon'_{xz} = -\frac{\theta'}{2}\left(y - 4b\cos\varphi\right). \quad (23)$$

The stresses σ'_{xz} and σ'_{yz} have the following expressions in the elastic area.

$$\sigma'_{xz} = -G\theta' y\left[1 + \left[1+\left(\frac{\tau_s}{4Gb\theta'}\right)^2\right]^{1/2} - \frac{\tau_s}{4Gb\theta'}\right], \quad (24)$$

$$\sigma'_{yz} = G\theta' x\left[1 - \left[1+\left(\frac{\tau_s}{4Gb\theta'}\right)^2\right]^{1/2} + \frac{\tau_s}{4Gb\theta'}\right]. \quad (25)$$

In the plastic area

$$\sigma'_{xz} = \tau_s\cos\varphi, \quad \sigma'_{yz} = \tau_s\sin\varphi. \quad (26)$$

Herewith, along all the cross section it holds the equation [6]:

$$y\sin\varphi + x\cos\varphi = 2b\sin 2\varphi$$

Note that the solution of (19)–(26) was obtained by V.V. Sokolovski [1] and is true provided $c \le a + b$, or.

$$\frac{\tau_s}{4Gb\theta'} + \left[1+\left(\frac{\tau_s}{4Gb\theta'}\right)^2\right]^{1/2} \le \frac{1}{2}\left(\frac{a}{b} - 1\right).$$

Residual Stresses and Strains of the Oval Bar: Define now residual stresses and strains that are preserved in the oval bar after removing the torque. and we will consider that in the unloading process in the bar there arise the areas of secondary plastic deformations. To this end, we will use the V.V.Moskvitin theorem on secondary plastic deformations [3]. According to this theorem, the residual stresses σ^0_{xz}, σ^0_{yz} strains ε^0_{xz}, ε^0_{yz} and displacements u^0_x, u^0_y, u^0_z of the twist θ may be determined by the formulas:

$$\sigma^0_{xz} = \sigma'_{xz} - \sigma^*_{xz}; \quad \sigma^0_{yz} = \sigma'_{yz} - \sigma^*_{yz};$$
$$\varepsilon^0_{xz} = \varepsilon'_{xz} - \varepsilon^*_{xz}; \quad \varepsilon^0_{yz} = \varepsilon'_{yz} - \varepsilon^*_{yz};$$
$$u^0_x = u'_x - u^*_x; \quad u^0_y = u'_y - u^*_y;$$
$$u^0_z = u'_z - u^*_z; \quad \theta^0 = \theta' - \theta^*. \quad (27)$$

Here $\sigma'_{xz}, \sigma'_{yz}, \varepsilon'_{xz}, \varepsilon'_{yz}, u'_x, u'_y, u'_z, \theta'$ are the stresses, strains, displacements, twists, respectively before the beginning of unloading, that are determined by the formulas (24), (25), (21)–(23), (1),(19), (20), (18). The quantities $\sigma^*_{xz}, \sigma^*_{yz}, \varepsilon^*_{xz}, \varepsilon^*_{yz}, u^*_x, u^*_y, u^*_z, \theta^*$ are the stresses, strains, displacements twists that hold in fictitious, oval bar at elasticoplastic torsion by the same torque M'. Unlike the considered bar the yield point at the shear of material of the fictitious bar is $2\varepsilon_s$. Between the quantities M' and θ^* it holds the relation (18) by changing τ_s by $2\tau_s$:

$$M' = \frac{2}{3}\pi\tau_s\left(2a^3 - 9ab^2 + 8b^3\right) - \frac{16}{3}\pi\tau_s b^3\left\{1 + 2\left(\frac{\tau_s}{2Gb\theta^*}\right)^3 - \right.$$

$$\left. - \left[1 - 2\left(\frac{\tau_s}{2Gb\theta^*}\right)^2\right]\left[1 + \left(\frac{\tau_s}{2Gb\theta^*}\right)^2\right]^{1/2}\right\}. \quad (28)$$

Herewith the elastic plastic boundary in the cross section of the fictitious, bar will be an ellipse with the semi-axes c^* and d^*:

$$\left.\begin{matrix} c^* \\ d^* \end{matrix}\right\} = 2b\left[\frac{\tau_s}{2Gb\theta'} + \sqrt{1 + \left(\frac{\tau_s}{2Gb\theta^*}\right)^2} \pm 1\right]. \quad (29)$$

In the elastic domain, the cross sections of the fictitious, bar will be:

$$\sigma^*_{xz} = -G\theta^* y\left\{1 + \left[1 + \left(\frac{\tau_s}{2Gb\theta^*}\right)^2\right]^{1/2} - \frac{\tau_s}{2Gb\theta^*}\right\}, \quad (30)$$

$$\sigma^*_{yz} = -G\theta^* x\left\{1 + \left[1 + \left(\frac{\tau_s}{2Gb\theta^*}\right)^2\right]^{1/2} + \frac{\tau_s}{2Gb\theta^*}\right\}, \quad (31)$$

$$\varepsilon^*_{xz} = -\frac{\theta^* y}{2}\left\{1 + \left[1 + \left(\frac{\tau_s}{2Gb\theta^*}\right)^2\right]^{1/2} - \frac{\tau_s}{2Gb\theta^*}\right\}, \quad (32)$$

$$\varepsilon^*_{yz} = \frac{\theta^* x}{2}\left\{1 + \left[1 + \left(\frac{\tau_s}{2Gb\theta^*}\right)^2\right]^{1/2} + \frac{\tau_s}{2Gb\theta^*}\right\}. \quad (33)$$

In the plastic domain of the cross section of the fictitious bar we have[7]:

$$\sigma^*_{xz} = 2\tau_s\cos\varphi, \qquad \sigma^*_{yz} = 2\tau_s\sin\varphi, \quad (34)$$

$$\varepsilon^*_{xz} = -\frac{\theta^*}{2}\left(y - 4b\cos\varphi\right), \qquad \varepsilon^*_{yz} = \frac{\theta^*}{2}x \quad (35)$$

In elastic and plastic domains the expressions will be written similarly for u^*_x, u^*_y land u^*.

We define the residual twist $\theta^0(M')$ by the formula $\theta^0(M') = \theta'(M', \tau_s, G, a, b) - \theta^*(M', \tau_s, G, a, b)$. Herewith the quantity θ' is found from the equation (18), the quantity θ^* from (28). The ellipse separating the area of elastic unloading and secondary plastic deformations are determined by the semi-axes c^0 and d^0 coinciding with the semi-axes c^* and d^* respectively.

$$\left.\begin{matrix} c^0 \\ d^0 \end{matrix}\right\} = 2b\left[\frac{\tau_s}{2Gb\left(\theta' - \theta^0\right)} + \sqrt{1 + \left(\frac{\tau_s}{2Gb\left(\theta' - \theta^0\right)}\right)^2} \pm 1\right] \quad (36)$$

Herewith the following condition should be fulfilled:

$$d^0 \le a - b \quad (37)$$

By decreasing the torque from M' to M_a, there will happen elastic unloading. By decreasing the torque from M_a to zero, the area of secondary plastic deformations extends from the point [0, ± (a - b)] to the centre of the cross section of the considered bar. Herewith, the external contours of the areas of secondary plastic deformations will the parts of the contour of the oval cross section with the semi-axes $a + b$, $a - b$ the internal contours will be the parts of the ellipse with the semi-axes c' and d', defined by the relations (36) (Fig. 1).

According to the theorem on secondary plastic deformations [3] and formula (16), the second plastic deformations in the unloading process will appear in the case if.

$$M' - M_a = \left(1 - 5\frac{b^2}{a^2} + \frac{51}{8}\frac{b^4}{a^4}\right)\frac{\pi a^5\tau_s}{(a-b)(a+2b)}.$$

Herewith, the condition of appearance of secondary plastic deformations at total unloading will be:

$$M' \ge \left(1 - 5\frac{b^2}{a^2} + \frac{51}{8}\frac{b^4}{a^4}\right)\frac{\pi a^5\tau_s}{(a-b)(a+2b)}. \quad (38)$$

It is clear that subject to condition (38), the condition (37) will be fulfilled.

At limit value $M_{\lim}$ of the torque M', the plastic area at initial torsion fills all the cross section and elastic kernel degenerates into some section whose length according to formula (17) will be $8b$. For the quantity $M_{\lim}$ according to formula (18), we have:

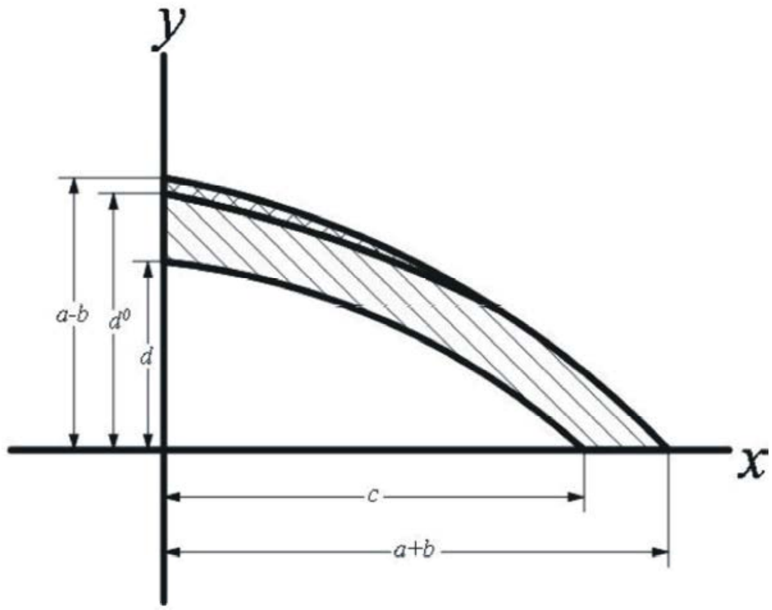

Fig. 1: Distribution of plastic deformations area in the quarter of an oval cross section bar at torsion and total unloading.

$$M_{\lim} = \frac{1}{3}\pi\tau_s\left(2a^3 - 9ab^2 + 8b^3\right). \quad (39)$$

Herewith, the value of the torque M' should be less than of $M_{\lim}$. Subject to (39) and inequality (38), the inequality $M' \le M_{\lim}$ fulfilled in the case if the oval cross section parameters a and b satisfies the condition:

$$\frac{a}{b} \ge 1{,}43. \quad (40)$$

Condition (40) holds subject to the inequality: $[(a + b)/(a - b)] \le 5{,}65$. Thus, in the bar made of ideally plastic material with cross section in the form of an ellipse, appearanc of secondary plastic deformations is possible only the case if the ratio of its great semi-axis to the small one doesn't exceed 5,65. Subject to this condition, the second plastic deformations at unloading will necessarily arise if the value of torque will satisfy condition (38).

Calculate now the residual stresses and strains. Herewith, we should bear in mind the existence of there different areas in the cross section (Fig. 2).

Area of secondary plastic deformations. According to formula (26) and (40) and formulas (23) and (35), in this area we have:

$$\sigma_{xz}^0 = -\tau_s\cos\varphi, \quad \sigma_{yz}^0 = -\tau_s\sin\varphi\,. \quad (41)$$

$$\varepsilon_{xz}^0 = -\frac{\theta^0}{2}\left(y - 4b\cos\varphi\right), \quad \varepsilon_{yz}^0 = \frac{\theta^0}{2}x. \quad (42)$$

The area between the ellipses with semi-axes (c, d) and (c^0, d^0). In this areas, according to formulas (26) and (30), (31) and also formulas (23) and (32), (33) we have:

$$\sigma_{xz}^0 = \delta_s\cos\varphi + G\theta^* y\left\{1 + \left[1 + \left(\frac{\tau_s}{2Gb\theta^*}\right)^2\right]^{1/2} - \frac{\tau_s}{2Gb\theta^*}\right\}, \quad (43)$$

$$\sigma_{yz}^0 = \delta_s\sin\varphi - G\theta^* x\left\{1 - \left[1 + \left(\frac{\tau_s}{2Gb\theta^*}\right)^2\right]^{1/2} + \frac{\tau_s}{2Gb\theta^*}\right\}, \quad (44)$$

$$\varepsilon_{xz}^0 = -\frac{\theta^0}{2}y + 2b\theta'\cos\varphi + \frac{\theta^* y}{2}\left\{\left[1 + \left(\frac{\tau_s}{2Gb\theta^*}\right)^2\right]^{1/2} - \frac{\tau_s}{2Gb\theta^*}\right\}, \quad (45)$$

$$\varepsilon_{yz}^0 = \frac{\theta^0}{2}x + \frac{\theta^*}{2}x\left\{\left[1 + \left(\frac{\tau_s}{2Gb\theta^*}\right)^2\right]^{1/2} - \frac{\tau_s}{2Gb\theta^*}\right\}. \quad (46)$$

The area inside the ellipse with semi-axes c and d. In this area, according to formulas (24), (25) and (30), (31) and also formulas (21), (22) and (32), (33) we have:

$$\sigma_{xz}^{0} = -G\theta^{0}y - G\theta' y\left\{\left[1+\left(\frac{\tau_s}{4Gb\theta'}\right)^2\right]^{1/2} - \frac{\tau_s}{4Gb\theta'}\right\} + G\theta^{*}y\left\{\left[1+\left(\frac{\tau_s}{2Gb\theta^{*}}\right)^2\right]^{1/2} - \frac{\tau_s}{2Gb\theta^{*}}\right\}, \quad (47)$$

$$\sigma_{yz}^{0} = G\theta^{0}y - G\theta' y\left\{\left[1+\left(\frac{\tau_s}{4Gb\theta'}\right)^2\right]^{1/2} - \frac{\tau_s}{4Gb\theta'}\right\} + G\theta^{*}x\left\{\left[1+\left(\frac{\tau_s}{2Gb\theta^{*}}\right)^2\right]^{1/2} - \frac{\tau_s}{2Gb\theta^{*}}\right\}, \quad (48)$$

$$\varepsilon_{xz}^{0} = -\frac{\theta^{0}}{2}y - \frac{\theta'}{2}y\left\{\left[1+\left(\frac{\tau_s}{4Gb\theta'}\right)^2\right]^{1/2} - \frac{\tau_s}{4Gb\theta'}\right\} + \frac{\theta^{*}}{2}y\left\{\left[1+\left(\frac{\tau_s}{2Gb\theta^{*}}\right)^2\right]^{1/2} - \frac{\tau_s}{2Gb\theta^{*}}\right\}, \quad (49)$$

$$\varepsilon_{yz}^{0} = \frac{\theta^{0}}{2}x - \frac{\theta'}{2}x\left\{\left[1+\left(\frac{\tau_s}{4Gb\theta'}\right)^2\right]^{1/2} - \frac{\tau_s}{4Gb\theta'}\right\} + \frac{\theta^{*}}{2}x\left\{\left[1+\left(\frac{\tau_s}{2Gb\theta^{*}}\right)^2\right]^{1/2} - \frac{\tau_s}{2Gb\theta^{*}}\right\}. \quad (50)$$

In the considered areas, the formulas of residual replacements may be written in the similar way. Formulas (41) – (50) are valid in the case when condition (38) and (40) are fulfilled, i.e. if at total unloading there appear secondary plastic deformations. But if one of the conditions (38) and (40), is not fulfilled, then at removing the torque there appears elastic unloading. Herewith, the residual stresses and strains may be calculated by A.A. Ilyushin's elastic unloading theorem [5]. According to this theorem, the residual sought-for values are determined by formulas (27). Herewith, the quantities $\sigma'_{xz}, \sigma'_{yz}, \varepsilon'_{xz}, \varepsilon'_{yz},\ u'_x, u'_y,\ u'_z,\ \theta'$ are determined from (24) - (26), (21) – (23), (1), (19), (20), (18). $\sigma^{*}_{xz}, \sigma^{*}_{yz}, \varepsilon^{*}_{xz}, \varepsilon^{*}_{yz},\ u^{*}_x, u^{*}_y,\ u^{*}_z, \theta^{*}$. But the quantities are the stresses, strains displacement s and twist existing in some fictitious bar of oval cross section at its elastic torsion by the torque that was applied to the considered bar before unloading. According to whaps has been said and on the base of formula (9) and (12) we write expressions for θ^{*}, $\sigma^{*}_{xz}, \sigma^{*}_{yz}, \varepsilon^{*}_{xz}$ and ε^{*}_{yz} :

$$M' = \left(1 - 5\frac{b^2}{a^2} + \frac{51}{8}\frac{b^4}{a^4}\right)\frac{\pi G a^4}{2}\theta^{*} \quad (51)$$

$$\sigma^{*}_{xz} = -\frac{G(a+2b)}{a}y\theta^{*}, \qquad \sigma^{*}_{yz} = \frac{G(a-2b)}{a}x\theta^{*}, \quad (52)$$

$$\varepsilon^{*}_{xz} = -\frac{a+2b}{2a}y\theta^{*}, \qquad \varepsilon^{*}_{yz} = \frac{a-2b}{2a}x\theta^{*}. \quad (53)$$

The residual twist will be: $\theta^0 = \theta' - \theta^*$ where θ' and θ^* are determined through the torque M' by formulas (18) and (51). The residual stresses and strains in the are a outside the ellipse with the semi-axes *c* and *d*, according to formulae (26), (52) and also formula (23), (53) will be:

$$\sigma_{xz}^{0} = \tau_s\cos\varphi + \frac{G(a+2b)}{a}y\theta^{*}; \quad \sigma_{yz}^{0} = \tau_s\sin\varphi - \frac{G(a-2b)}{a}x\theta^{*};$$

$$\varepsilon_{xz}^{0} = -\frac{\theta^{0}}{2}y + 2b\theta'\cos\varphi + \frac{b}{a}y\theta^{*}.$$

$$\varepsilon_{yz}^{0} = \frac{\theta^{0}}{2}x + \frac{b}{a}x\theta^{*}.$$

In the area inside the ellipse with the semi-axes *c* and *d* according to formulas (24), (25), (52) and also formulas (21), (22), (53), we have:

$$\sigma_{xz}^{0} = -G\theta^{0}y - G\theta' y\left\{\left[1+\left(\frac{\tau_s}{4Gb\theta'}\right)^2\right]^{1/2} - \frac{\tau_s}{4Gb\theta'}\right\} + 2G\frac{b}{a}y\theta^{*},$$

$$\sigma_{yz}^{0} = G\theta^{0}x - G\theta' x\left\{\left[1+\left(\frac{\tau_s}{4Gb\theta'}\right)^2\right]^{1/2} - \frac{\tau_s}{4Gb\theta'}\right\} + 2G\frac{b}{a}x\theta^{*}.$$

$$\varepsilon_{xz}^{0} = -\frac{\theta^{0}}{2}y - \frac{\theta'}{2}y\left\{\left[1+\left(\frac{\tau_s}{4Gb\theta'}\right)^2\right]^{1/2} - \frac{\tau_s}{4Gb\theta'}\right\} + \frac{b}{a}y\theta^{*}.$$

$$\varepsilon_{yz}^{0} = \frac{\theta^{0}}{2}x - \frac{\theta'}{2}x\left\{\left[1+\left(\frac{\tau_s}{4Gb\theta'}\right)^2\right]^{1/2} - \frac{\tau_s}{4Gb\theta'}\right\} + \frac{b}{a}x\theta^{*}.$$

Fatigue Failure of Oval Bar at Pulsating Torsion: Now let's consider a fatigue failure of the considered bar under pulsating torsion. We will use cyclic damage and the cyclic durability conditions. According to [5], the cyclic damage condition has the following form:

$$\frac{N_1\left(\varepsilon_t^0\left(N^1\right)\right)}{N_0\left(\varepsilon_t^0\left(N^1\right)\right)-N_1\left(\varepsilon_t^0\left(N^1\right)\right)}=\int_0^{N'}\frac{dk}{N_0\left(\varepsilon_t^0(k)\right)-N_1\left(\varepsilon_t^0(k)\right)}. \tag{54}$$

The cyclic durability condition is written as follows:

$$\frac{N_0\left(\varepsilon_t^0\left(N^1\right)\right)}{N_0\left(\varepsilon_t^0(N_*)\right)-N_1\left(\varepsilon_t^0(N_*)\right)}=\int_0^{N_*}\frac{dk}{N_0\left(\varepsilon_t^0(k)\right)-N_1\left(\varepsilon_t^0(k)\right)} \tag{55}$$

In relations (54) and (55) ε^0_t is the intensity of residual deformations in the κ-the loading circle; N' is the number of loading cycles at which in the material the damage accumulation process begins for $\varepsilon_t^0=\varepsilon_t^0(k)$; N_* is the number of loading cycles to failure for $\varepsilon_t^0=\varepsilon_t^0(k)$; $N_1=N_1\left(\varepsilon_t^0\right)$ and $N_0=N_0\left(\varepsilon_t^0\right)$ are of the functions experimentally defined for each material, herewith N_1 and N_0 are the loading cycles before appearance of damages in the experimental sample to failure. In relations (54) and (55), N' and N_* are the sought for quantities.

The functions N_1 and N_0 may be approximated in the form:

$$N_1=A_1\left(\varepsilon_t^0\right)^{\alpha_1};\quad N_0=A_0\left(\varepsilon_t^0\right)^{\alpha_0} \tag{56}$$

When processing experimental data of the paper [4], for the steel of mark 45 the following values were obtained:

$$\alpha_1=\alpha_0=-2{,}5;\quad A_1=3{,}9\cdot10^{-4};\quad A_0=8{,}8\cdot10^{-4}$$

Herewith $N_1/N_0 = B = 0{,}4.5$ In [5] it was mentioned that the ratio $N_1/N_0 = B = const$ holds for several materials. Proceeding from this fact, conditions (54) and (55) are transformed to the form:

$$\int_0^{N_*}\frac{dk}{N_0\left(\varepsilon_t^0(k)\right)}=1;\quad \int_0^{N'}\frac{dk}{N_0\left(\varepsilon_t^0(k)\right)}=B. \tag{57}$$

Use conditions (57) for determining the amount of pulsating torsion cycles to the first damage and failure of the considered oval cross section bar. To this end, we define the intensity of residual deformations $\varepsilon^0_t(k)$, where κ is the current amount of torsion.

Since the bar's material is ideally plastic, then the residual deformations for any κ–cycle of total unloading will be the same as in the first total unloading. This conclusion is valid also in the case if in the first total unloading there arise secondary plastic deformations.

We calculate the intensity of residual deformations by the formula:

$$\varepsilon_t^0=\frac{2}{\sqrt{3}}\left(\varepsilon_{xz}^0+\varepsilon_{yz}^0\right)^{1/2}. \tag{58}$$

In the oval bar the section $x = 0$ is dangerous for failure. From physical reasoning's it follows that failure of the oval bar begins at the points (0, a - b); (0,-(a-b). Furthermore, existence of secondary plastic deformations at total unloading accelerates the metal failure process. Proceeding from this fact, by calculating the intensity of residual deformations, we use formulas (42). For $x = 0$, $y = a - b$, we have:

$$\varepsilon_{xz}^0=-\frac{\theta^0}{2}(a+3b);\qquad \varepsilon_{yz}^0=0.$$

Taking into account the last relation in (58), we get:

$$\varepsilon_t^0=\frac{\theta^0}{\sqrt{3}}(a+3b) \tag{59}$$

Formula (59) holds in any κ-th cycle of the pulsating torsion, i.e. the quantity ε^0_t is independent of the number of pulsating torsion. Herewith from (57) it follows that the quantities N_* and N' coincide with the quantities N_0 and BN_0, respectively. Consequently, we have:

$$N'=BA_0\left[\frac{\theta^0}{\sqrt{3}}(a+3b)\right]^{\alpha_0};\quad N_*=A_0\left[\frac{\theta^0}{\sqrt{3}}(a+3b)\right]^{\alpha_0}.$$

At the above mentioned values α_0, B, A_0 for the steel of mark 45 and also at the values $\frac{M'}{\tau_s a^3}=1,\quad \frac{G\theta^0 a}{\tau_s}=1{,}5$ and $\frac{a}{b}=6{,}5$ for the quantity N' and N_* we get the following values: $N' = 1{,}2.\ 10^5$ cycle, $N_* = 2{,}67.\ 10^5$ cycle.

CONCLUSION

Conditions of appearance of secondary plastic deformations at total unloading after preliminary elastico-plastic torsion of oval cross section bar were determined.

The residual stresses and deformations at elastic unloading plastic deformations, that are preserved in the oval bar at elactico -plastic torsion, are determined.

The numbers of the cycles of pulsating elastico-plastic torsions at which the first damage appears and failure begins were determined.

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