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Arbitrary Type D Vacuum Spacetimes and Lanczos Potential

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Abstract: For any type D empty 4-space, in the canonical null tetrad, we determine the Andersson-Edgar's potential for the Lanczos spintensor.

Key words: Lanczos generator · Newman-Penrose formalism · Type D vacuum 4-spaces · Bianchi identities · Andersson-Edgar potential · Weyl-Lanczos equations

INTRODUCTION

Here we consider an arbitrary empty spacetime, type D in the Petrov classification [1-3], with a canonical null tetrad $(l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$ [4] such that:

$$\psi_r = 0, \quad r \neq 2, \quad \psi_2 \neq 0, \quad \kappa = \sigma = \lambda = \nu = 0, \tag{1}$$

In terms of the Newman-Penrose (NP) formalism [1, 5-9]. Besides, the Bianchi identities acquire the form:

$$D\psi_2 = 3\rho\psi_2, \quad \Delta\psi_2 = -3\mu\psi_2, \quad \delta\psi_2 = 3\tau\psi_2, \quad \bar{\delta}\psi_2 = -3\pi\psi_2. \tag{2}$$

On the other hand, the Lanczos spintensor $K_{\mu\nu\alpha}$ [10, 11] generates the conformal tensor via the equations [8, 12-14]:

$$\begin{aligned}
\psi_0 &= 2f[\delta\Omega_0 - D\Omega_4 + (-\bar{\alpha} - 3\beta + \bar{\pi})\Omega_0 + 3\sigma\Omega_1 + (\bar{\rho} - 3\varepsilon - \bar{\varepsilon})\Omega_4 - 3\kappa\Omega_5],\\
2\psi_1 &= f[\Delta\Omega_0 + 3\delta\Omega_1 + \bar{\delta}\Omega_4 - 3D\Omega_5 - (3\gamma + \bar{\gamma} + 3\mu - \bar{\mu})\Omega_0 + 3(-\bar{\alpha} - \beta + \bar{\pi} + \tau)\Omega_1 + 6\sigma\Omega_2 +\\
&(3\alpha - \bar{\beta} + 3\pi + \bar{\tau})\Omega_4 + 3(\varepsilon - \bar{\varepsilon} - \rho + \bar{\rho})\Omega_5 - 6\kappa\Omega_6],\\
\psi_2 &= f[\Delta\Omega_1 + \delta\Omega_2 - \bar{\delta}\Omega_5 - D\Omega_6 - \nu\Omega_0 - (2\mu - \bar{\mu} + \gamma + \bar{\gamma})\Omega_1 + (-\bar{\alpha} + \beta + \bar{\pi} + 2\tau)\Omega_2 + \sigma\Omega_3 +\\
&\lambda\Omega_4 + (\alpha - \bar{\beta} + 2\pi + \bar{\tau})\Omega_5 - (\varepsilon + \bar{\varepsilon} - \bar{\rho} + 2\rho)\Omega_6 - \kappa\Omega_7],\\
2\psi_3 &= f[3\Delta\Omega_2 + \delta\Omega_3 - 3\bar{\delta}\Omega_6 - D\Omega_7 - 6\nu\Omega_1 + 3(\bar{\mu} - \mu - \bar{\gamma} + \gamma)\Omega_2 + (-\bar{\alpha} + 3\beta + 3\tau + \bar{\pi})\Omega_3 +\\
&6\lambda\Omega_5 + 3(-\alpha - \bar{\beta} + \pi + \bar{\tau})\Omega_6 - (3\varepsilon + \bar{\varepsilon} - \bar{\rho} + 3\rho)\Omega_7],\\
\psi_4 &= 2f[\Delta\Omega_3 - \bar{\delta}\Omega_7 - 3\nu\Omega_2 + (\bar{\mu} + 3\gamma - \bar{\gamma})\Omega_3 + 3\lambda\Omega_6 + (-3\alpha - \bar{\beta} + \bar{\tau})\Omega_7],
\end{aligned} \tag{3}$$

where the Ω_r, = r = 0,...,7 are the NP components of $K_{\alpha\beta\gamma}$ and $f(x^j)$ must be determined for the given geometry; with (1) and (2) we can obtain the following solution of the Weyl-Lanczos equations (3) [15, 16]:

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$$\Omega_k = 0, \quad k = 0,3,4,7, \qquad f = -\psi_2^{2/3},$$
$$\Omega_1 = \frac{\rho}{2}\psi_2^{-2/3}, \quad \Omega_2 = \frac{\pi}{2}\psi_2^{-2/3}, \quad \Omega_5 = \frac{\tau}{2}\psi_2^{-2/3}, \quad \Omega_6 = \frac{\mu}{2}\psi_2^{-2/3}, \tag{4}$$

with the important participation of spin coefficients associated to the canonical null tetrad.

The Lanczos spinor [13, 17-21] is given by:

$$L_{ABCD} = l_A l_B l_C(\Omega_0 l_D - \Omega_4 o_D) + (l_A l_B o_C + (o_A * l_B)l_c)(-\Omega_1 l_D + \Omega_5 o_D) +$$
$$(o_A o_B l_C + (o_A * l_B)o_C)(\Omega_2 l_D - \Omega_6 o_D) + o_A o_B o_C(-\Omega_3 l_D + \Omega_7 o_D), \tag{5}$$

In terms of the spinors associated to the null tetrad:

$$l^\mu \leftrightarrow o^A o^B, \quad n^\mu \leftrightarrow l^A l^B, \quad m^\mu \leftrightarrow o^A l^B, \quad \bar{m}^\mu \leftrightarrow l^A o^B; \tag{6}$$

Besides andersson-Edgar [22-24] proved that any Lanczos spinor can be generated via the relation:

$$L_{ABCD} = \nabla^E D^T{}_{ABCD}, \quad T_{ABCE} = T_{(ABC)E}, \tag{7}$$

Such that:

$$T_{ABCE} = l_A l_B l_C(\Lambda_0 l_E - \Lambda_4 o_E) + (l_A l_B o_C + (o_A * l_B)l_c)(-\Lambda_1 l_E + \Lambda_5 o_E) +$$
$$(o_A o_B l_C + (o_A * l_B)o_C)(\Lambda_2 l_E - \Lambda_6 o_E) + o_A o_B o_C(-\Lambda_3 l_E \Lambda_7 o_E). \tag{8}$$

Therefore, (5), (7) and (8) imply the following NP equations:

$$\begin{aligned}
\Omega_0 &= \bar{\delta}\Lambda_0 - D\Lambda_4 + (\pi - 4\alpha)\Lambda_0 + 3\rho\Lambda_1 + (2\varepsilon + \rho)\Lambda_4 - 3\kappa\Lambda_5,\\
\Omega_1 &= \bar{\delta}\Lambda_1 - D\Lambda_5 - \lambda\Lambda_0 + (\pi - 2\alpha)\Lambda_1 + 2\rho\Lambda_2 + \pi\Lambda_4 + \rho\Lambda_5 - 2\kappa\Lambda_6,\\
\Omega_2 &= \bar{\delta}\Lambda_2 - D\Lambda_6 - 2\lambda\Lambda_1 + \pi\Lambda_2 + \rho\Lambda_3 + 2\pi\Lambda_5 + (\rho - 2\varepsilon)\Lambda_6 - \kappa\Lambda_7,\\
\Omega_3 &= \bar{\delta}\Lambda_3 - D\Lambda_7 - 3\lambda\Lambda_2 + (2\alpha + \pi)\Lambda_3 + 3\pi\Lambda_6 + (\rho - 4\varepsilon)\Lambda_7,\\
\Omega_4 &= \Delta\Lambda_0 - \delta\Lambda_4 + (\mu - 4\gamma)\Lambda_0 + 3\tau\Lambda_1 + (2\beta + \tau)\Lambda_4 - 3\sigma\Lambda_5,\\
\Omega_5 &= \Delta\Lambda_1 - \delta\Lambda_5 - \nu\Lambda_0 + (\mu - 2\gamma)\Lambda_1 + 2\tau\Lambda_2 + \mu\Lambda_4 + \tau\Lambda_5 - 2\sigma\Lambda_6,\\
\Omega_6 &= \Delta\Lambda_2 - \delta\Lambda_6 - 2\nu\Lambda_1 + \mu\Lambda_2 + \tau\Lambda_3 + 2\mu\Lambda_5 + (\tau - 2\beta)\Lambda_6 - \sigma\Lambda_7,\\
\Omega_7 &= \Delta\Lambda_3 - \delta\Lambda_7 - 3\nu\Lambda_2 + (2\gamma + \mu)\Lambda_3 + 3\mu\Lambda_6 + (\tau - 4\beta)\Lambda_7,
\end{aligned} \tag{9}$$

If we select the values:

$$\Lambda_r = 0, \quad r \neq 2,5, \quad \Lambda_2 = \Lambda_5 = \frac{1}{10}\psi_2^{-2/3}, \tag{10}$$

then with (9) we reproduce the expressions (4), that is, the Andersson-Edgar's potential is given by:

$$T_{ABCE} = \frac{1}{10}\psi_2^{-2/3}[(l_A l_B o_C + (o_A * l_B)l_C)o_E + (o_A o_B l_C + (o_A * l_B)o_C)l_E], \tag{11}$$

for any type D empty spacetime in the canonical null tetrad.

The results (7), (10) and (11) are equivalent to the tensor relation:

$$K_{\mu\nu\alpha} = U_{\mu\nu\alpha\beta}{}^{;\beta}, \tag{12}$$

Such that $U_{\mu\nu\alpha\beta}$ has the algebraic symmetries of the Weyl tensor $C_{\mu\nu\alpha\beta}$ and (12) implies the following NP equations:

$$\begin{aligned}
\Omega_0 &= D\gamma_1 - \bar{\delta}\gamma_0 + (4\alpha - \pi)\gamma_0 - 2(2\rho + \varepsilon)\gamma_1 + 3\kappa\gamma_2, \\
\Omega_1 &= D\gamma_2 - \bar{\delta}\gamma_1 + \lambda\gamma_0 + 2(\alpha - \pi)\gamma_2 - 3\rho\gamma_2 + 2\kappa\gamma_3, \\
\Omega_2 &= D\gamma_3 - \bar{\delta}\gamma_2 + 2\lambda\gamma_1 - 3\pi\gamma_2 + 2(\varepsilon - \rho)\gamma_3 + \kappa\gamma_4, \\
\Omega_3 &= D\gamma_4 - \bar{\delta}\gamma_3 + 3\lambda\gamma_2 - 2(2\pi + \alpha)\gamma_3 + (4\varepsilon - \rho)\gamma_4, \\
\Omega_4 &= \delta\gamma_1 - \Delta\gamma_0 + (4\gamma - \mu)\gamma_0 - 2(2\tau + \beta)\gamma_1 + 3\sigma\gamma_2, \\
\Omega_5 &= \delta\gamma_2 - \Delta\gamma_1 + \nu\gamma_0 + 2(\gamma - \mu)\gamma_1 + 3\tau\gamma_2 + 2\sigma\gamma_3, \\
\Omega_6 &= \delta\gamma_3 - \Delta\gamma_2 + 2\nu\gamma_1 - 3\mu\gamma_2 + 2(\beta - \tau)\gamma_3 + \sigma\gamma_4, \\
\Omega_7 &= \delta\gamma_4 - \Delta\gamma_3 + 3\nu\gamma_2 - 2(2\mu - \gamma)\gamma_3 + (4\beta - \tau)\gamma_4,
\end{aligned} \tag{13}$$

where the γ_r, $r = 0,1,\ldots,4$ are the NP components of $U_{\mu\nu\alpha\beta}$, similar to the ψ_r for $C_{\mu\nu\alpha\beta}$; then (13) and the values:

$$\gamma_r = 0, \ \ r \neq 2, \qquad \gamma_2 = -\frac{1}{10}\psi_2^{-2/3}, \tag{14}$$

Generate the expressions (4), that is, the Lanczos potential for any type D vacuum geometry has the structure (12) in the corresponding canonical null tetrad, such that:

$$\begin{aligned}
U_{\mu\nu\alpha\beta} + i * U_{\mu\nu\alpha\beta} &= -\frac{1}{5}\psi_2^{-2/3}(M_{\mu\nu}M_{\alpha\beta} + V_{\mu\nu}U_{\alpha\beta} + U_{\mu\nu}V_{\alpha\beta}), \\
V_{ab} = l_a x\, m_b, \quad U_{ab} &= \bar{m}_a x\, n_b, \ \ M_{ab} = m_a\ x\ \bar{m}_b + n_a\ x\ l_b.
\end{aligned} \tag{15}$$

The relation (12) occurs in the Gödel cosmological model [25].

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