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Modified Laplace Transform and Ordinary Differential Equations with Variable Coefficients

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Abstract: This work presents a computational calculation to solve ordinary differential equations with variable coefficients by using the modified version of Laplace and Sumudu transform. Modified transform, whose crucial properties and theorems is displayed in this paper. Explanatory examples are displayed to appear the productivity of its appropriateness.

Key words: Laplace transform · Modified Laplace transform (MLT) · Sumudu transform · Differential equations with variable coefficients

INTRODUCTION

Integral transform method is broadly used to solve the several differential equations with the initial values or boundary conditions, [5-19]. Within the writing there are various integral transforms and broadly utilized in material science, astronomy as well as in engineering. In arrange to solve the differential equations, the integral transform are broadly used and in this way there are a few works on the theory and application of integral transform such as the Laplace, Fourier, Mellin, and Hankel.

Elzaki transform [1, 2 3, 4, 14, 17, 18, 19], which is a modified Laplace and Sumudu transform, [1] has been shown to solve effectively, easily and accurately a large class of linear differential equations. Elzaki transform was successfully applied to integral equations, partial differential equations [2], ordinary differential equations with variable coefficients [4] and system of all these equations.

The reason of this paper is to illuminate differential equations with variable coefficients which were not fathomed by Sumudu transform and the trouble of understanding them with Laplace transform; this implies that Sumudu transform and Laplace transform fizzled to fathom these sorts of differential equations.

Recently Tarig M. Elzaki [1, 2 3, 4], modified Laplace transform (MLT) that is defined by the integral equation:

$$E\left[f(t)\right]=k^2\int_0^\infty f(kt)e^{-t}dt=k\int_0^\infty f(t)e^{-\frac{t}{k}}dt=T(k),c \tag{1}$$

Theorems: Recall the following theorems that were given by Tarig Elzaki [11, 2 3, 4] where they discussed modified Laplace transform (MLT) of the derivatives.

Theorem 1: Let $T(k)$ is (MLT) of $f(t)$ $[E(f(t)) = T(k)$ then:

$$(i)\quad \mathrm{E}\left[f'(t)\right]=\frac{T(k)}{k}-kf(0)\qquad (ii)\quad \mathrm{E}\left[f''(t)\right]=\frac{T(k)}{k^2}-f(0)-kf'(0)$$

$$(iii)\quad \mathrm{E}\left[f^{(n)}(t)\right]=\frac{T(k)}{k^n}-\sum_{i=0}^{n-1}u^{2-n+i}f^{(i)}(0)$$

Proof:

(i) $\mathrm{E}\left[f'(t)\right]=k\int_0^\infty f'(t)\, e^{\frac{-t}{k}}\,dt$ Integrating by parts to find that:

$$\mathrm{E}\left[f'(t)\right]=\frac{T(k)}{k}-kf(0)$$

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(*ii*) Let $g(t) = f'(t)$, then:

$$\mathrm{E}\left[g'(t)\right]=\frac{1}{k}\ \mathrm{E}\left[g(t)\right]-kg(0)$$

We find that, by using (*i*):

$$\mathrm{E}\left[f''(t)\right] = \frac{T(k)}{k^2}-f(0)-kf'(0) \tag{2}$$

We can prove (*iii*) by mathematical induction.

where that Sumudu transform of derivatives is given by:

$$S[f'(t)]=\frac{1}{v}\left[F(v)-f(0)\right], \text{ and}$$

$$S[f''(t)]=\frac{1}{v^2}\left[F(v)-f(0)-vf'(0)\right] \tag{3}$$

Theorem 2:

Let $f(t) \in A = \{f(t) \mid \exists\, M, k_1, k_2 > 0, \textit{such that} \| f(t)| < Me^{|t|/ki}$, $\textit{if}\ (t) \in (-1)^j \times [0,\infty)\}$

With Laplace transform $F(s)$, Then modified Laplace transform (MLT) $T(k)$ of $f(t)$ is given by: $T(k)=k\,F\left(\frac{1}{k}\right)$

Proof:

Let: $f(t) \in A$, then for, $k_1 < k < k_2$ $T(k)=k^2\int_0^\infty e^{-t}f(kt)\,dt$

Let w - kt then we have:

$$T(k)=k^2\int_0^\infty e^{-\frac{w}{k}}f(w)\frac{dw}{k}=k\int_0^\infty e^{-\frac{w}{k}}f(w)dw=k\,F\left(\frac{1}{k}\right).$$

Also we have that T (1) = F(1) so that both the modified Laplace transform (MLT) and Laplace transform must coincide at $k = s = 1$.

Theorem 3: Let $T(k)$ be (MLT) of the function $f(t)$ in A, then (MLT) of the function $tf(t)$ is given by:

$$\mathrm{E}\left[tf(t)\right]=k^2\frac{d}{du}T(k)-kT(k)$$

Proof: The function $tf(t)$ is in A, is *ce f* (*t*) so: and integrating by parts we find that:

$$\frac{d}{dk}T(k)=T'(k)\ =\frac{d}{dk}\int_0^\infty ke^{-\frac{t}{k}}f(t)dt=\int_0^\infty\frac{\partial}{\partial k}\left[ke^{-\frac{t}{k}}f(t)\right]dt$$

$$=\int_0^\infty\frac{1}{k}e^{-\frac{t}{k}}\left(t\,f(t)\right)dt+\int_0^\infty e^{-\frac{t}{k}}f(t)\ dt=\ \frac{1}{k^2}\mathrm{E}\left[t\,f(t)\right]+\frac{1}{k}\ \mathrm{E}\left[f(t)\right]$$

Then we have: $\mathrm{E}\left[t\ f(t)\right]=\ k^2\frac{d}{dk}T(k)-kT(k)$

In the general cases we can extension Theorem 3 as,

$$(i)\,\mathrm{E}\left[t\,f'(t)\right]=k^2\frac{d}{dk}\left[\frac{T(k)}{k}-kf(0)\right]-k\left[\frac{T(k)}{k}-kf(0)\right]$$

$$(ii)\ \mathrm{E}\left[t^2f'(t)\right]=k^4\frac{d^2}{dk^2}\left[\frac{T(k)}{k}-kf(0)\right]$$

$$(iii)\ \mathrm{E}\left[t\,f''(t)\right]=k^2\frac{d}{dk}\left[\frac{T(k)}{k^2}-f(0)-kf'(0)\right]-k\left[\frac{T(k)}{k}-f(0)-kf'(0)\right]$$

$$(iv)\ \mathrm{E}\left[t^2f''(t)\right]=\ k^4\frac{d^2}{dk^2}\left[\frac{T(k)}{k^2}-f(0)-kf'(0)\right]$$

The proof of these equations is easy, by using theorem 3.

And Sumudu transform of these is given by:

$$(i)\,S\left[tf(t)\right]=v^2\frac{d}{dv}F(v)+vF(v)$$

$$(ii)\,S\left[t^2f(t)\right]=v^4\frac{d^2}{du^2}F(v)+4v^3\frac{d}{dv}F(v)+2v^2F(v) \tag{4}$$

where $F(v)$ is the Sumudu transform of $f(t)$.

The Main Problem: In this section we introduce the second order ordinary differential equations with variable coefficients.

Consider the second order ordinary differential equations with variable coefficients,

$$f_1\,(t)\,y''+f_2\,(t)\,y'+f_3\,(t)\,y=g(t) \tag{5}$$

With the initial conditions:

$y(0) = \alpha$, $y'\,(0) = \beta$

where, $f_1(t) = a_1t^2 + b_1t + c_1$, $f_2(t) = a_2t^2 + b_2t + c_2$, $f_3(t) = a_3t^2 + b_3t + c_3$,

And $g(t)$ is the function of t.

By applying the (MLT) to the Eq. (5), and theorems 1 and 3, we have:

$$k^4 a_1 \frac{d^2}{dk^2}\left[\frac{T(k)}{k^2} - f(0) - kf'(0)\right] + k^2 b_1 \frac{d}{dk}\left[\frac{T(k)}{k^2} - f(0) - kf'(0)\right] - kb_1\left[\frac{T(k)}{k} - f(0) - kf'(0)\right] +$$
$$c_1\left[\frac{T(k)}{k^2} - f(0) - kf'(0)\right] + k^4 a_2 \frac{d^2}{dk^2}\left[\frac{T(k)}{k} - kf(0)\right] + k^2 b_2 \frac{d}{dk}\left[\frac{T(k)}{k} - kf(0)\right] - kb_2\left[\frac{T(k)}{k} - kf(0)\right]$$
$$+c_2\left[\frac{T(k)}{k} - kf(0)\right] + a_3 k^4 \frac{d^2}{dk^2}\left[T(k)\right] + b_3 k^2 \frac{d}{dk}\left[T(k)\right] - b_3 k T(k) + c_3 T(k) = G(k)$$

Applying the initial conditions to obtain:

$$\left[k^2 a_1 + k^3 a_2 + a_3 k^4\right] T''(k) + \left[b_1 - 4a_1 k - 2a_2 k^2 + b_2 k + b_3 k^2\right] T'(k)$$
$$+\left[6a_1 - \frac{3}{k} b_1 + \frac{c_1}{k^2} + 2a_2 k + \frac{c_2}{k} + b_3 k - 2b_2 + c_3\right] T(k) = G(k) + c_1\alpha - b_1 k\alpha + c_1\beta k + c_2\alpha k \tag{6}$$

This is the second order differential equations.
Again we applying the Laplace transform to Eq. (5), and making use of the initial conditions to find that:

$$\left[s^2 a_1 + s a_2 + a_3\right] F''(s) + \left[2a_2 - b_3 + 4a_1 s - b_2 s - b_1 s^2\right] F'(s)$$
$$+\left[2a_1 - 2b_1 s + c_1 s^2 - b_2 + c_2 s + c_3\right] F(s) = G(s) + c_1\alpha s + c_1\beta + c_2\alpha - b_1\alpha \tag{7}$$

Lastly if we take Sumudu transform of Eq. (5), and making use of the initial conditions and the formula of Sumudu transform, we obtain that:

$$\left[a_1 u^2 + a_2 u^3 + a_3 u^4\right] 3F''(u) + \left[b_1 + 2a_2 u^2 + b_2 u + 4a_3 u^2 + b_3 u^2\right] F'(u)$$
$$+\left[6 - 6a_1 - \frac{b_1}{u} + \frac{c_1}{u^2} + \frac{c_2}{u} + 2a_3 u^2 + b_3 u + c_3\right] F(u)$$
$$= G(u) - \frac{\alpha b_1}{u} + \frac{\alpha c_1}{u^2} + \frac{\beta c_1}{u} - 4\alpha a_2 u - \alpha b_2 + \frac{\alpha c_2}{u} \tag{8}$$

Applications: In this section we will apply the above three integral transforms and equations (6), (7), (8) to some differential equations with variable coefficients, to show the effectiveness of the modified Laplace transform and its preference over the Laplace and Sumudu transform.

Example (4-1): Consider the second order differential equation with variable coefficients,

$$ty'' + 2y' = -\cos t \tag{9}$$

With the conditions,

$$\lim_{t\to 0} y(t) = 0 \Leftrightarrow y(0) = 0\,,\ \lim_{t\to 0} y'(t) = -\frac{1}{2} \Leftrightarrow y'(0) = -\frac{1}{2} \tag{10}$$

Solution: To solve this equation we must apply the equations (6), (7), (8) respectively such that;
$a_1 = 0,\ b_1 = 1,\ c_1 = 0,\ a_2 = 0,\ b_2 = 0,\ c_2 = 2,\ a_3 = b_3 = c_3 = 0$

Solution by Modified Laplace Transform:
Equation (6) becomes,

$$T'(k)-\frac{1}{k}T(k)=\frac{-k^2}{1+k^2},$$

This is linear differential equations and we can solve it as;

$$T(k)=-k\left[\frac{k^2}{2}-\frac{k^4}{4}+\frac{k^6}{6}-\frac{k^8}{8}+....\right]$$

Taking the inverse modified Laplace transform to find:

$$y(t)=-\frac{t}{2!}+\frac{t^3}{4!}-\frac{t^5}{6!}+\frac{t^7}{8!}+....=\frac{\cos t-1}{t}$$

This is the exact solution of Eq. (9).

Solution by Laplace Transform:
Equation (7) becomes,

$$F'(s)=\frac{1}{s(s^2+1)}=\sum_{i=0}^{\infty}(-1)^i s^{2i-1} \Rightarrow F(s)=\sum_{i=0}^{\infty}\frac{(-1)^i}{2i}s^{2i} \quad (10)$$

This equation is not easy to solve because it's difficult to find the inverse Laplace transform, this means that Eq. (9) cannot or difficult to solve by Laplace transform.

Solution by Sumudu Transform:
Equation (8) becomes,

$$F'(v)+\left(6+\frac{1}{v}\right)F(v)=\frac{-1}{1+v^2},$$

this is a linear differential equation, but it's difficult to solve it by Sumudu transform.

Example (4-2): Consider the first order differential equation with variable coefficients,

$$t^2y'+2ty=\sinh t \quad (11)$$

With the condition,

$$\lim_{t\to 0} y(t)=\frac{1}{2} \Leftrightarrow y(0)=\frac{1}{2}$$

Solution:
From Eq. (11), we have:

$$a_1=b_1=c_1=0, a_2=1, b_2=c_2=0, a_3=0, b_3=2, c_3=0, \alpha=\frac{1}{2}$$

Solution by Modified Laplace Transform:
Equation (6) becomes,

$$k^3T''(k)=\frac{k^3}{1-k^2} \Rightarrow T''(k)=\sum_{i=0}^{\infty}k^{2i},$$

The solution of this equation is;

$$T(k)=\sum_{i=0}^{\infty}\frac{k^{2i+2}}{(2i+1)(2i+2)}$$

Taking the inverse modified Laplace transform to obtain the exact solution in the form:

$$y(t)=\sum_{i=0}^{\infty}\frac{t^{2i}}{(2i+2)!}=\frac{1}{2!}+\frac{t^2}{4!}+\frac{t^4}{6!}+\frac{t^6}{8!}+.... \Rightarrow y(t)=\frac{\cosh t-1}{t^2}$$

If we use the same method that is used in example (4-1), then Eqs. (7), (8) becomes respectively as:

$$F''(s)=\frac{1}{s(s^2-1)}=\sum_{i=0}^{\infty}s^{2i-1} \Rightarrow F(s)=s+\frac{s^3}{6}+\frac{s^5}{20}+\frac{s^7}{42}+.... \quad (12)$$

$$v^3F''(v)+4v^2F'(v)+(6+2v)F(v)=\frac{1}{1-v^2}-\frac{5}{2}v \quad (13)$$

Again it's difficult to find the inverse Laplace transform of Eq. (12), and then it's difficult to find the solution of Eq. (12) by Laplace transform.

Equation (13) cannot be solved because it has become more difficult than equation (11), and therefore equation (11) cannot be solved by Sumudu transform.

CONCLUSION

In this paper, we have presented the modified version of Sumudu and Laplace transform, to be specific modified Laplace transform (MLT) for solving differential equations with variable coefficients which were not solved by Sumudu transform and the trouble of fathoming them with Laplace transform. It has been appeared that modified Laplace transform (MLT) it could be an exceptionally compelling strategy for solving initial value problems compared with Sumudu transform and Laplace transform. In a large space the accurate convergence of the modified Laplace transform (MLT) will be talked about on the coming studies.

Availability of Data and Materials: Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing Interests: The authors declare that they have no competing interests.

Authors' Contributions: The authors read and agreed the final manuscript.

Appendix:

Modified Laplace Transform (MLT) of Some Functions

$f(t)$	$E[f(t)] = T(k)$
1	k^2
t	k^3
t^n	$n!k^{n+2}$
$t^{a-1}/\Gamma(a), a>0$	k^{a+1}
e^{at}	$\frac{k^2}{1-ak}$
te^{at}	$\frac{k^3}{(1-ak)^2}$
$\frac{t^{n-1}e^{at}}{(n-1)!},\ n=1,2,....$	$\frac{k^{n+1}}{(1-ak)^n}$
$\sin at$	$\frac{ak^3}{1+a^2k^2}$
$\cos at$	$\frac{k^2}{1+a^2k^2}$
$\sinh at$	$\frac{ak^3}{1-a^2k^2}$
$\cosh at$	$\frac{ak^2}{1-a^2k^2}$
$e^{at}\sin bt$	$\frac{bk^3}{(1-ak)^2+b^2k^2}$
$e^{at}\cos bt$	$\frac{(1-ak)k^2}{(1-ak)^2+b^2k^2}$
$t\sin at$	$\frac{2ak^4}{1+a^2k^2}$
$J_0(at)$	$\frac{k^2}{\sqrt{1+ak^2}}$
$H(t-a)$	$k^2e^{-\frac{a}{k}}$
$\delta(t-a)$	$ke^{-\frac{a}{k}}$

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