World Engineering & Applied Sciences Journal 10 (1): 08-12, 2019
ISSN 2079-2204

DOI: 10.5829/idosi.weasj.2019.08.12

Rotations via Quaternions

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Abstract: We study how generate orthogonal 4x4-matrices using a quaternionic triple product, which leads in natural manner to Dirac matrices and the analysis of rotations in three and four dimensions.

Key words: Spatial rotations · Dirac matrices · Lorentz transformations · Quaternions

INTRODUCTION

The quaternionic units obey the algebra [1-5]:

$$\mathbf{I}^2 = \mathbf{J}^2 = \mathbf{K}^2 = -1. \qquad \mathbf{IJK} = -1. \tag{1}$$

which allows realize the product:

$$\tilde{\mathbf{F}} = \mathbf{p}\ \mathbf{F}, \tag{2.a}$$

with:

$$\mathbf{F} = F_1\ \mathbf{I} + F_2\ \mathbf{J} + F_3\ \mathbf{K} + F_4, \tag{2.b}$$

$$\mathbf{p} = p_1\ \mathbf{I} + p_2\ \mathbf{J} + p_3\ \mathbf{K} + p_4, \tag{2.c}$$

then (2.a) acquires the matrix form [6]:

$$\begin{pmatrix} \tilde{F}_1 \\ \tilde{F}_2 \\ \tilde{F}_3 \\ \tilde{F}_4 \end{pmatrix} = \begin{pmatrix} p_4 & -p_3 & p_2 & p_1 \\ p_3 & p_4 & -p_1 & p_2 \\ -p_2 & p_1 & p_4 & p_3 \\ -p_1 & -p_2 & -p_3 & p_4 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix} \equiv P \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix}. \tag{3}$$

The square of the magnitude of **F** is defined by:

$$|\mathbf{F}|^2 = \mathbf{F}\bar{\mathbf{F}} = \bar{\mathbf{F}}\mathbf{F} = {F_1}^2 + {F_2}^2 + {F_3}^2 + {F_4}^2, \tag{4.a}$$

such that:

$$\bar{\mathbf{F}} = -F_1\ \mathbf{I} - F_2\ \mathbf{J} - F_3\ \mathbf{K} + F_4\ , \tag{4.b}$$

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and we note that for any **A** and **B**:

$$\overline{\mathbf{AB}} = \overline{\mathbf{B}}\,\overline{\mathbf{A}}\,. \qquad (4.c)$$

Thus from (2.a, 3, 4.a,c):

$$\left|\tilde{\mathbf{F}}\right|^2 = \mathbf{p}\,|\mathbf{F}|^2\,\overline{\mathbf{p}} = \mathbf{p}\,\overline{\mathbf{p}}\,|\mathbf{F}|^2, \quad \det P = (\mathbf{p}\,\overline{\mathbf{p}})^2, \qquad (5.a)$$

therefore if **p** is unitary:

$$\mathbf{p}\,\overline{\mathbf{p}} = p_1{}^2 + p_2{}^2 + p_3{}^2 + p_4{}^2 = 1, \qquad \det P = 1, \qquad (5.b)$$

then the magnitude of **F** is constant:

$$\tilde{F}_1{}^2 + \tilde{F}_2{}^2 + \tilde{F}_3{}^2 + \tilde{F}_4{}^2 = F_1{}^2 + F_2{}^2 + F_3{}^2 + F_4{}^2, \qquad (6.a)$$

which is equivalent to:

$$(\tilde{F}_1 \quad \tilde{F}_2 \quad \tilde{F}_3 \quad \tilde{F}_4)\begin{pmatrix}\tilde{F}_1\\ \tilde{F}_2\\ \tilde{F}_3\\ \tilde{F}_4\end{pmatrix} = (F_1 \quad F_2 \quad F_3 \quad F_4)\begin{pmatrix}F_1\\ F_2\\ F_3\\ F_4\end{pmatrix}, \qquad (6.b)$$

hence (3) and (6.b) imply the orthogonal character of the transformation:

$$P^T P = I_{4X4}\,. \qquad (7)$$

Similarly, the product:

$$\tilde{\mathbf{F}} = \mathbf{F}\,\mathbf{q}\,, \qquad (8.a)$$

has the matrix representation:

$$\begin{pmatrix}\tilde{F}_1\\ \tilde{F}_2\\ \tilde{F}_3\\ \tilde{F}_4\end{pmatrix} = \begin{pmatrix} q_4 & q_3 & -q_2 & q_1\\ -q_3 & q_4 & q_1 & q_2\\ q_2 & -q_1 & q_4 & q_3\\ -q_1 & -q_2 & -q_3 & q_4\end{pmatrix}\begin{pmatrix}F_1\\ F_2\\ F_3\\ F_4\end{pmatrix} \equiv Q\begin{pmatrix}F_1\\ F_2\\ F_3\\ F_4\end{pmatrix}, \qquad (8.b)$$

and (6.a) is valid if **q** is unitary, then Q is orthogonal with det $Q = 1$

The cases (2.a) and (8.a) can be unified via a triple quaternionic product:

$$\tilde{\mathbf{F}} = \mathbf{p}\,\mathbf{F}\,\mathbf{q}\,, \qquad (9.a)$$

that is:

$$\begin{pmatrix}\tilde{F}_1\\ \tilde{F}_2\\ \tilde{F}_3\\ \tilde{F}_4\end{pmatrix} = D_{4X4}\begin{pmatrix}F_1\\ F_2\\ F_3\\ F_4\end{pmatrix}, \qquad D = P\,Q, \qquad (9.b)$$

thus $|\tilde{\mathbf{F}}|^2 = (\mathbf{p}\overline{\mathbf{p}})(\mathbf{q}\overline{\mathbf{q}})|\mathbf{F}|^2$ and (6.a) is verified for $|\mathbf{p}| = |\mathbf{q}| = 1$ with P and Q orthogonal matrices, therefore:

$$D^T D = I, \qquad \det D = 1. \tag{9.c}$$

Now we shall realize applications of (9.a,b,c): The Sec. 2 shows that D reproduces the sixteen Dirac matrices [7] if **p** and **q** coincide with the quaternionic units. The Sec 3 considers the case of special relativity because D generates Lorentz transformations [8-10] if $\mathbf{q} = \overline{\mathbf{p}}^*$. The Sec. 4 is dedicated to 3-rotations when **p** is real and unitary, that is, $|\mathbf{p}| = 1$, $\mathbf{p} = \mathbf{p}^*$ and $\mathbf{q} = \overline{\mathbf{p}}$.

Dirac Matrices: In relativistic quantum mechanics are important the following sixteen 4x4-matrices [7]:

$$I, \quad \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^0\gamma^5 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \gamma^r = \begin{pmatrix} 0 & \sigma_r \\ -\sigma_r & 0 \end{pmatrix}, r = 1,2,3, \tag{10}$$

$\sigma^{0r} = -\sigma^{r0} = i\begin{pmatrix} 0 & \sigma_r \\ \sigma_r & 0 \end{pmatrix}, \sigma^{jk} = -\sigma^{kj} = \begin{pmatrix} \sigma_l & 0 \\ 0 & \sigma_l \end{pmatrix},$ (jkl) is a cyclic permutation of (123),

with the Cayley-Sylvester-Pauli matrices [1, 11-13]:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad i = \sqrt{-1}. \tag{11}$$

Now in (9.a) we can select to **p** and **q** as the quaternionic units, for example, if $\mathbf{p} = 1$, $\mathbf{q} = \mathbf{K}$, then from (9.b) we obtain that $D = i\,\sigma^{31}$; similarly, if $\mathbf{p} = \mathbf{I}$, $\mathbf{q} = \mathbf{J}$, therefore $D = -\gamma^1\gamma^5$, etc. Thus, this process allows construct the Table [14]:

$\mathbf{p} \setminus \mathbf{q}$	1	$\mathbf{I}$	$\mathbf{J}$	$\mathbf{K}$
1	I	γ^1	$-\gamma^3$	$i\,\sigma^{31}$
$\mathbf{I}$	σ^{02}	$-\gamma^3\gamma^5$	$-\gamma^1\gamma^5$	$-\gamma^5$
$\mathbf{J}$	$\gamma^0\gamma^5$	σ^{32}	σ^{12}	$i\,\gamma^2$
$\mathbf{K}$	$-i\,\gamma^2\gamma^5$	$i\,\sigma^{03}$	$i\,\sigma^{01}$	γ^0

(12)

If w denotes any matrix into this Table, then is simple prove the properties:

$$W^{-1} = W^T, \qquad W^* = W, \qquad W^T = \pm W, \tag{13}$$

that is, all matrices are real, orthogonal and symmetric or antisymmetric.

Lorentz Transformations: In Minkowski spacetime [8] the linear Lorentz transformations connect the coordinates of two reference frames in uniform relative motion [c is the velocity of light in vacuum]:

$$\begin{pmatrix} i\tilde{x} \\ i\tilde{y} \\ i\tilde{z} \\ c\tilde{t} \end{pmatrix} = D \begin{pmatrix} ix \\ iy \\ iz \\ ct \end{pmatrix}, \tag{14.a}$$

which in geometrical terms corresponds to 4-dimensional rotation:

$$(c\tilde{t})^2 - \tilde{x}^2 - \tilde{y}^2 - \tilde{z}^2 = (ct)^2 - x^2 - y^2 - z^2, \tag{14.b}$$

in according with the postulates of special relativity.

If the quaternion (2.b) is selected as:

$$\mathbf{F} = ix\ \mathbf{I} + iy\ \mathbf{J} + iz\ \mathbf{K} + ct, \tag{15}$$

then (6.a) and (9.b) reproduce (14.a,b) and (9.a) implies:

$$i\tilde{x}\ \mathbf{I} + i\tilde{y}\ \mathbf{J} + i\tilde{z}\ \mathbf{K} + c\tilde{t} = \mathbf{p}\ (ix\ \mathbf{I} + iy\ \mathbf{J} + iz\ \mathbf{K} + ct)\ \mathbf{q}, \tag{16.a}$$

where we can apply the operations * and (4.b,c) to obtain:

$$i\tilde{x}\ \mathbf{I} + i\tilde{y}\ \mathbf{J} + i\tilde{z}\ \mathbf{K} + c\tilde{t} = \overline{\mathbf{q}}^*\ (ix\ \mathbf{I} + iy\ \mathbf{J} + iz\ \mathbf{K} + ct)\ \overline{\mathbf{p}}^*, \tag{16.b}$$

whose comparison with (16.a) gives $\overline{\mathbf{q}}^* = \mathbf{p}$, that is:

$$\mathbf{q} = \overline{\mathbf{p}}^*, \qquad |\mathbf{p}| = 1. \tag{17}$$

Thus (9.a) acquires the structure [8, 15-26]:

$$\tilde{\mathbf{F}} = \mathbf{p}\ \mathbf{F}\ \overline{\mathbf{p}}^*, \qquad \mathbf{p}\ \overline{\mathbf{p}} = 1, \tag{18}$$

then with (15) we can generate Lorentz transformations verifying (9.c). For example, if we select:

$$\mathbf{p} = -i\ \text{senh}\left(\frac{\tau}{2}\right)\mathbf{K} + \cosh\left(\frac{\tau}{2}\right), \tag{19.a}$$

we obtain the known relations:

$$\tilde{x} = x, \quad \tilde{y} = y, \quad \tilde{z} = \gamma(z - vt), \quad \tilde{t} = \gamma\left(t - \frac{v}{c^2}z\right), \quad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \quad \tanh\tau = \frac{v}{c} < 1, \tag{19.b}$$

connecting to two observers with motion in the *z*-direction and relative velocity *v*.

Rotations in Three Dimensions: Here we consider Lorentz transformations such that:

$$\tilde{t} = t, \tag{20}$$

which corresponds to 3-rotations. If we apply (20) into (16.a), then it is possible to eliminate *t* when **pq** = 1, thus from (5.b) and (17):

$$\mathbf{p} = \mathbf{q}^* \tag{21}$$

that is, the Lorentz matrices generate spatial rotations if in (9.a) **p** is unitary and real. Then (14.a) gives the expressions:

$$\begin{pmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\end{pmatrix} = R\begin{pmatrix}x\\ y\\ z\end{pmatrix}, \quad R\,R^T = I, \quad \det R = 1, \qquad \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 = x^2 + y^2 + z^2. \tag{22.a}$$

From (9.a), (17) and (21) is immediate to deduce the following structure of *R* reported in the literature [24, 27-34]:

$$R = \begin{pmatrix} 1-2(p_2^2+p_3^2) & 2(p_1p_2-p_3p_4 & 2(p_1p_3+p_2p_4) \\ 2(p_1p_2+p_3p_4 & 1-2(p_1^2+p_3^2) & 2(p_2p_3-p_1p_4 \\ 2(p_1p_3-p_2p_4 & 2(p_1p_4+p_2p_3) & 1-2(p_1^2+p_2^2) \end{pmatrix}. \qquad (22.b)$$

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