World Engineering & Applied Sciences Journal 10 (1): 04-07, 2019
ISSN 2079-2204

DOI: 10.5829/idosi.weasj.2019.04.07

Applications of Bell Polynomials

J. López-Bonilla, R. López-Vázquez and S. Vidal-Beltrán

ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso,
Col. Lindavista CP 07738, CDMX, México

Abstract: We show that certain recurrence relation connected with the complete Bell polynomials allows deduce the identities of Batir, Connon, Shen and Spiess involving the Stirling numbers of the first kind.

Key words: Bell polynomials · Stirling numbers · Harmonic sums

INTRODUCTION

In [1] was proved that the recurrence relation:

$$m\, a_m + \sum_{l=1}^{m} s_l\, a_{m\text{-}l} = m\, a_m + s_1\, a_{m-1} + \cdots + s_{m-1}\, a_1 + s_m = 0, \quad m = 1, 2, \ldots, \quad a_0 = 1, \tag{1}$$

is equivalent to the following expression in terms of the complete Bell polynomials [1-10]:

$$k!\, a_k = Y_k(-0!\ s_1, -1!\ s_2, -2!\ s_3, -3!\ s_4, \ldots, \text{-}(k-2)!\ s_{k-1}, \text{-}(k-1)!\ s_k), \quad k = 0, 1, 2, \ldots \tag{2}$$

where:

$$Y_m(x_1, x_2, \ldots, x_m) = \begin{vmatrix} \binom{m-1}{0} x_1 & \binom{m-1}{1} x_2 & \cdots & \binom{m-1}{m-2} x_{m-1} & \binom{m-1}{m-1} x_m \\ -1 & \binom{m-2}{0} x_1 & \cdots & \binom{m-2}{m-3} x_{m-2} & \binom{m-2}{m-2} x_{m-1} \\ 0 & -1 & \cdots & \binom{m-3}{m-4} x_{m-3} & \binom{m-3}{m-3} x_{m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \binom{1}{0} x_1 & \binom{1}{1} x_2 \\ 0 & 0 & \cdots & -1 & \binom{0}{0} x_1 \end{vmatrix}, \tag{3}$$

that is:

$$Y_0 = 1,\ \ Y_1 = x_1,\ \ Y_2 = x_1^2 + x_2,\ \ Y_3 = x_1^3 + 3x_1\, x_2 + x_3\,,\ \ Y_4 = x_1^4 + 6\, x_1^2\, x_2 + 4\, x_1\, x_3 + 3\, x_2^2 + x_4,$$
$$Y_5 = x_1^5 + 10\, x_1^3\, x_2 + 10\, x_1^2\, x_3 + 15\, x_1\, x_2^2 + 5\, x_1\, x_4 + 10\, x_2\, x_3 + x_5,\ \ \ldots \tag{4}$$

therefore:

$$a_1 = \text{-}s_1\,, \qquad 2!\ a_2 = (s_1)^2\text{-}s_2\,, \qquad 3!\ a_3 = \text{-}(s_1)^3 + 3\, s_1\, s_2 - 2\, s_3\,,$$

$$4!\ a_4 = (s_1)^4 - 6\,(s_1)^2\, s_2 + 8\, s_1\, s_3 + 3\,(s_2)^2 - 6\, s_4\,,$$

$$5!\ a_5 = \text{-}(s_1)^5 + 10\,(s_1)^3\, s_2 - 20\,(s_1)^2\, s_3 - 15\, s_1 (s_2)^2 + 30\, s_1\, s_4 + 20\, s_2\, s_3 - 24\, s_5,\ \ldots \tag{5}$$

Corresponding Author: J. López-Bonilla, ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México.

In Sec. 2 we employ (1) and (2) to deduce several identities for the Stirling numbers of the first kind [2, 3, 7, 11-13].

Stirling Numbers: We know the following relation for the Stirling numbers [3, 8, 10, 14, 15]:

$$S_n^{(r+1)} = \frac{(-1)^{n-r-1}(n-1)!}{r!} Y_r\left(0!\,H_{n-1}^{(1)}, -1!\,H_{n-1}^{(2)}, 2!\,H_{n-1}^{(3)}, -3!\,H_{n-1}^{(4)}, \ldots, (-1)^{r-1}(r-1)!\,H_{n-1}^{(r)}\right), \quad (6)$$

for $0 \leq r \leq n-1$ and $n \geq 1$which acquires the structure (2) with the definitions:

$$a_k = \frac{(-1)^{n-k-1}}{(n-1)!} S_n^{(k+1)}, \quad s_k = (-1)^k H_{n-1}^{(k)}, \quad a_0 = \frac{(-1)^{n-1}}{(n-1)!} S_n^{(1)} = \frac{(-1)^{n-1}}{(n-1)!} (-1)^{n-1}(n-1)! = 1, \quad (7)$$

then (1) and (7) imply the Shen's identity [15, 16]:

$$-r\, S_n^{(r+1)} = \sum_{k=0}^{r-1} S_n^{(r-k)}\ H_{n-1}^{(k+1)}, \quad (8)$$

where [17, 18]:

$$H_n^{(m)} = \sum_{j=1}^{n} \frac{1}{j^m}, \quad H_n^{(0)} = n, \quad n \geq 1, \quad H_k^{(m)} = 0, \ k \leq 0. \quad (9)$$

In [19] was obtained the expression:

$$k!\ S_n^{(n-k)} = Y_k(-0!\ H_{n-1}^{(-1)}, -1!\ H_{n-1}^{(-2)}, -2!\ H_{n-1}^{(-3)}, \ldots, -(k-2)!\ H_{n-1}^{(-(k-1))}, -(k-1)!\ H_{n-1}^{(-k)}), \quad (10)$$

such that $H_{n-1}^{(-j)} = \sum_{q=1}^{n-1} q^j$, $j = 1, \ldots, k.$ From (4) and (10) are immediate the values [7, 20]:

$$S_n^{(n-1)} = -\binom{n}{2} = -\frac{n(n-1)}{2}, \quad S_n^{(n-2)} = 3\binom{n}{4} + 2\binom{n}{3} = \frac{n(n-1)(n-2)(3n-1)}{24}, \quad (11)$$

If we define the quantities:

$$a_k = S_n^{(n-k)}, \quad s_k = H_{n-1}^{(-k)}, \quad k = 1, \ldots, n, \quad a_0 = S_n^{(n)} = 1, \quad (12)$$

then (10) adopts the structure (2), thus (1) and (12) imply the Spiess identity [17]:

$$\sum_{k=r-n}^{r-1} S_n^{(r-k)}\ H_{n-1}^{(k+1)} = 0, \quad \forall\ r\ \varepsilon\ \mathbb{Z} \quad (13)$$

which is a particular case of [17]:

$$\sum_{j=1}^{n-1} \binom{j}{p} H_j^{(r+1)} = \binom{n}{p+1} H_{n-1}^{(r+1)} - \frac{1}{(p+1)!} \sum_{j=1}^{p+1} S_{p+1}^{(j)}\ H_{n-1}^{(r+1-j)}, \quad n \geq 1, \ p \geq 0, \ r\ \varepsilon\ \mathbb{Z}, \quad (14)$$

for $p = n-1$.

Batir [18] realized a study of the quantities:

$$S_n(m) \equiv \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{k^m}, \quad m, n \geq 1, \quad (15)$$

and, for example, he obtains the expressions:

$$S_n(1) = H_n\,, \qquad S_n(2) = \tfrac{1}{2}\left(H_n^2 + H_n^{(2)}\right), \qquad S_n(3) = \tfrac{1}{6}\left(H_n^3 + 3\,H_n\,H_n^{(2)} + 2\,H_n^{(3)}\right), \tag{16}$$

$$S_n(4) = \tfrac{1}{24}\left[H_n^4 + 6\,H_n^2\,H_n^{(2)} + 8\,H_n\,H_n^{(3)} + 3\,(H_n^{(2)})^2 + 6\,H_n^{(4)}\right],$$

$$S_n(5) = \tfrac{1}{120}\left[H_n^5 + 10\,H_n^3\,H_n^{(2)} + 20\,H_n^2\,H_n^{(3)} + 15\,H_n\,(H_N^{(2)})^2 + 30\,H_n\,H_n^{(4)} + 20\,H_n^{(2)}H_n^{(3)} + 24\,H_n^{(5)}\right], \tag{17}$$

The relations (16) were also deduced by Flajolet-Sedgewick [21].

In [8, 15] was proved the property:

$$S_n(m) = \tfrac{1}{m!}\,Y_m\left(H_n, 1!\,H_n^{(2)},\ 2!\,H_n^{(3)},\ \dots, (m-1)!\,H_n^{(m)}\right), \tag{18}$$

which takes the form (2) if introduce the definitions:

$$a_m = S_n(m), \qquad a_0 = S_n(0) = 1, \qquad s_k = -H_n^{(k)}, \tag{19}$$

then (1) gives the following recurrence relation:

$$\sum_{k=1}^{m} S_n(m\text{-}k)\ H_n^{(k)} = m\,S_n(m), \qquad m \geq 1. \tag{20}$$

Hence the expressions (16) and (17) can be obtained from (18) or (20).

REFERENCES

1. López-Bonilla, J., S. Vidal-Beltrán and A. Zúñiga-Segundo, 2018. Characteristic equation of a matrix via Bell polynomials, Asia Mathematika, 2(2): 49-51.
2. Riordan, J., 1968. Combinatorial identities, John Wiley and Sons, New York.
3. Comtet, L., 1974. Advanced combinatorics, D. Reidel Pub., Dordrecht, Holland.
4. Zave, D.A., 1976. A series expansion involving the harmonic numbers, Inform. Process. Lett., 5(3): 75-77.
5. Johnson, W.P., 2002. The curious history of Faà di Bruno's formula, The Math. Assoc. of America, 109: 217-234.
6. Chen, X. and W. Chu, 2009. The Gauss ${}_2F_1(1)$-summation theorem and harmonic number identities, Integral Transforms and Special Functions, 20(12): 925-935.
7. Quaintance, J. and H.W. Gould, 2016. Combinatorial identities for Stirling numbers, World Scientific, Singapore.
8. López-Bonilla, J., R. López-Vázquez, S. Vidal-Beltrán, 2018. Bell polynomials, Prespacetime, 9(5): 451-453.
9. López-Bonilla, J., S. Vidal-Beltrán and A. Zúñiga-Segundo, 2018. On certain results of Chen and Chu about Bell polynomials, Prespacetime Journal, 9(7): 584-587.
10. López-Bonilla, J., S. Vidal-Beltrán and A. Zúñiga-Segundo, 2018. Some applications of complete Bell polynomials, World Eng. & Appl. Sci. J., 9(3): 89-92.
11. López-Bonilla, J., J. Yaljá Montiel-Pérez and S. Vidal-Beltrán, 2018. On the Janjic's definition of Stirling numbers, European J. Appl. Sci., 10(3): 84.
12. Barrera-Figueroa, V., J. López-Bonilla, R. López-Vázquez and S. Vidal-Beltrán, 2018. On Stirling numbers, World Scientific News, 98: 228-232.
13. López-Bonilla, J., J. Yaljá Montiel-Pérez and O. Salas-Torres, 2018. Some identities for Stirling numbers, Comput. Appl. Math. Sci., 31(2): 13-14.
14. Kölbig, K.S., 1994. The complete Bell polynomials for certain arguments in terms of Stirling numbers of the first kind, J. Comput. Appl. Math., 51: 113-116.
15. Connon, D.F., 2010. Various applications of the (exponential) complete Bell polynomials, http://arxiv.org/ftp/arxiv/papers/1001/1001.2835.pdf 16 Jan 2010

16. Shen, L.C., 1995. Remarks on some integrals and series involving the Stirling numbers and ζ(n), Trans. Amer. Math. Soc., 347: 1391-1399.
17. Spiess, J., 1990. Some identities involving harmonic numbers, Maths. of Comput., 55(192): 839-863.
18. Batir, N., 2017. On some combinatorial identities and harmonic sums, Int. J. Number Theory, 13(7): 1695-1709.
19. López-Bonilla, J., R. López-Vázquez and S. Vidal-Beltrán, 2019. Stirling numbers and Bell polynomials, Prespacetime Journal, 10(2):
20. Knuth, D.E., 2003. Selected papers on Discrete Mathematics, CSLI Lecture Notes, No. 106.
21. P. Flajolet, P. and R. Sedgewick, 1995. Mellin transforms and asymptotics: Finite differences and Rice's integrals, Theor. Computer Sci., 144: 101-124.