World Engineering & Applied Sciences Journal 10 (1): 01-03, 2019
ISSN 2079-2204

DOI: 10.5829/idosi.weasj.2019.01.03

Finite Binomial Expansion of Gregory-Newton

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Abstract: We employ the Gregory-Newton's polynomial interpolation to prove the Newton's binomial theorem and deduce relations involving harmonic and Stirling numbers & also Laguerre polynomials.

Key words: Gregory-Newton interpolation · Laguerre polynomials · Harmonic and Stirling numbers · Equidistant Lagrangian interpolation · Binomial theorem

INTRODUCTION

The function can be interpolated with a polynomial of degree $(n-1)$ in the equidistant data points $[x_1, x_2, x_3,..., x_{n-1}, x_n] = [0,1,2,..., n\text{-}2, n-1]$ via the finite binomial Gregory [1-3] - Newton [4] expansion [5, 6]:

$$f(x) = \sum_{k=0}^{n-1} \binom{x}{k} \Delta^k f(0) + \eta_n(x), \tag{1}$$

with the simple differences of Euler (1755):

$$\Delta^k f(0) = \sum_{r=0}^{k} \binom{k}{r} (-1)^{k-r} f(r), \tag{2}$$

and the remainder of Lagrangian interpolation [6, 7]:

$$\eta_n(x) = \binom{x}{n} f^{(n)}(\bar{x}), \quad \bar{x}\varepsilon[0, n-1], \; \eta_n(j) = 0,$$
$$j = 0,1,2,..., n-1, \quad n \geq 1. \tag{3}$$

In Sec. 2 we apply (1) to several functions to obtain the Newton binomial theorem [4, 8, 9], relations involving Stirling and harmonic numbers [10-13] and also the Laguerre polynomials [14-21].

Applications of Gregory-Newton Expansion: Here we consider applications of (1) to various functions, for example:

- $f(x) = x^n$. Therefore, (2) implies the Euler's expression [11]:

$$\Delta^k f(0) = (-1)^k \sum_{r=0}^{k} (-1)^r \binom{k}{r} r^n = k!\, S_n^{[k]}, \tag{4}$$

with the participation of the Stirling numbers of the second kind $S_n^{[k]}$ [10, 11, 13]. Besides, $f^{(n)}(x) = n!$, thus $\eta_n(x) = n!\binom{x}{n}$ and from (1):

$$x^n = \sum_{k=0}^{n-1} k! \binom{x}{k} S_n^{[k]} + n! \binom{x}{n} = \sum_{k=0}^{n} \binom{x}{k} k! S_n^{[k]}, \tag{5}$$

because, $S_n^{[n]} = 1$.

- $f(x) = \frac{t^x}{x!}$. Then (2) gives the Lanczos relation [6]:

$$\Delta^k f(0) = (-1)^k (1 - f)^k = (-1)^k L_k(t), \tag{6}$$

where we employ the Rainville's notation $f^r = f(r)$ and $L_k(t)$ denotes a Laguerre polynomial [14-21]. Thus (1) implies the expression:

$$\frac{t^x}{x!} = \sum_{k=0}^{n-1} (-1)^k \binom{x}{k} L_k(t) + \eta_n(x), \tag{7}$$

which for $x = n\text{-}1 = m$ allows deduce the property:

$$\frac{t^m}{m!} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} L_k(t). \tag{8}$$

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- $f(x) = (1+\frac{a}{b})^x$. Therefore $\Delta^k f(0) = (\frac{a}{b})^k$ and from (1):

$$(1+\frac{a}{b})^x = \sum_{k=0}^{n-1} \binom{x}{k} \frac{a^k}{b^k} + \eta_n(x), \tag{9}$$

then for $x = n$-1 $= m$ we obtain the Newton's binomial theorem [4, 8, 9]:

$$(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k}. \tag{10}$$

- $f(x) = \frac{(x+y+p)!}{(x+p)!(y+p)!}, y \geq 0,\ p, x\ \varepsilon(-\infty,\infty)\ \ \therefore\ \ \Delta^k f(0) = \frac{k!}{(p+k)!}\binom{y}{k}$ and from (1) with $x = n$-1 $= m =$ 1,2,...:

$$\frac{(y+p+m)!}{(y+p)!(p+m)!} = \sum_{k=0}^{m} \frac{k!}{(p+k)!}\binom{y}{k}\binom{m}{k}, \tag{11}$$

which is equivalent to:

$$\frac{\binom{y+p+m}{p+m}}{\binom{y+p}{p}} = \sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{p+k}{p}}\binom{y}{k}. \tag{12}$$

where we can use $p = N =$ 0,1,2,... and after to realize $[\frac{d}{dy}(12)]_{y=0}$ to deduce the following identity involving the harmonic numbers [10-12]:

$$H_{m+N} - H_N = \sum_{k=1}^{m} \frac{1}{N+k} = \sum_{k=1}^{m} \frac{(-1)^{k+1}}{k} \frac{\binom{m}{k}}{\binom{N+k}{N}}, \tag{13}$$

because [11]:

$$[\frac{d}{dy}\binom{y+n}{n}]_{y=0} = H_n,\ \ [\frac{d}{dy}\binom{y}{k}]_{y=0} \frac{}{k!}S_k^{(1)} = \frac{(-1)^{k+1}}{k}, \tag{14}$$

with $S_k^{(j)}$ a Stirling number of the first kind.

REFERENCES

1. Gregory, J., 1670. Letter to John Collins, 23 Nov.
2. Dehn, M. and E. Hellinger, 1943. Certain mathematical achievements of James Gregory, Am. Math. Monthly, 50(3): (1943) 149-163.
3. J. Beery, J. and J.A. Stedall, 2009. Thomas Harriot's doctrine of triangular numbers: the 'Magisteria Magna', European Math. Soc., Zürich, Switzerland.
4. Newton, I., 1687. Principia, London (1687) Lemma 5, Book III.
5. Lanczos, C., 1988. Applied analysis, Dover, New York
6. Lanczos, C., 1996. Linear differential operators, SIAM, Philadelphia, USA.
7. Guerrero-Moreno, I., J. López-Bonilla and S. Vidal-Beltrán, 2019. On the remainder of Lagrangian interpolation formula, American-Eurasian J. Sci. Res., 4(1): 1-3.
8. Coolidge, J.L., 1949. The story of the binomial theorem, Am. Math. Monthly, 56(3): 147-157.
9. Whiteside, D.T., 1961. Newton's discovery of the general binomial theorem, Math. Gazette, 45(353): 175-180.
10. Srivastava, H.M. and J. Choi, 2012. Zeta and q-zeta functions and associated series and integrals, Elsevier, London.
11. J. Quaintance, J. and H.W. Gould, 2016. Combinatorial identities for Stirling numbers, World Scientific, Singapore.
12. Carvajal, B., J. López-Bonilla, R. López-Vázquez, 2017. On harmonic numbers, Prespacetime Journal, 8(4): 484-489.
13. Barrera-Figueroa, V., J. López-Bonilla and R. López-Vázquez, 2017. On Stirling numbers of the second kind, Prespacetime Journal, 8(5): 572-574.
14. Laguerre, E.N., 1879. Sur l'intégrale $\int_x^\infty \frac{e^{-x}}{x}dx$, Bull. Soc. Math. France, 7: 72-81.
15. Abramowitz, M. and I.A. Stegun, 1972. Handbook of mathematical functions, Wiley and Sons, New York Chap., pp: 22.
16. López-Bonilla, J., A. Lucas-Bravo, S. Vidal-Beltrán, 2005. Integral relationship between Hermite and Laguerre polynomials: Its application in quantum mechanics, Proc. Pakistan Acad. Sci., 42(1): 63-65.

17. López-Bonilla, J. and A. Lucas-Bravo, 2008. A note on the finite Taylor series applied to Laguerre polynomials, Educatia Matematica, 4(1): 53-55.
18. Bucur, A., B. Carvajal and J. López-Bonilla, 2009. Laguerre polynomials: Their Laplace transform via equidistant interpolation, J. Sci. Res. (India), 53: 257-258.
19. R. Cruz-Santiago, R., J. López-Bonilla, S. Yáñez-San Agustín, 2017. A note on the Laguerre polynomials, Prespacetime Journal, 8(4): 511-512.
20. J. López-Bonilla, J., R. López-Vázquez, V.M. Salazar del Moral, 2017. On some identities for the Laguerre polynomials, Prespacetime Journal, 8(10): 1251-1254.
21. S. Álvarez, S., J. López-Bonilla, R. López-Vázquez, 2017. On the roots of Legendre, Laguerre and Hermite polynomials, Borneo J. Sci. & Tech., 38(2): 41-45.