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Crossed Product of Inverse Semigroups by Topological Action

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Abstract: The notion of topologically free partial action of a group on a C^*-algebra is generalized to a *topological action* of an inverse semigroup on a C^*-algebra. Also, partial crossed product of a C^*-algebra and a group by a partial action is generalized to the crossed product $A\times_\alpha S$ in which α is the *action* of the unital inverse semigroup S on the C^*-algebra A. *Invariant ideal* under the action of an inverse semigroup and its associated *quotient action* are discussed.

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INTRODUCTION

In the last decade, the theory of crossed product of C^*-algebras by semigroups of endomorphisms has been developing successfully. This theory is a generalization of the theory of crossed product of C^*-algebras by groups of automorphisms, which is a well-established area of operator algebras. Another variant of crossed product which is different from the group actions by automorphisms, involves partial actions by partial automorphisms and was introduced by R. Exel [3] to study circle actions on C^*-algebras. Crossed products by partial actions of discrete groups were treated by MacClanahan [8] and were subsequently characterized from differed perspectives by Quigg and Raeburn [10], Exel [5], Quigg [9] and Exel, Laca and Quigg [2].

The well-established notion of the crossed product of a C^*-algebra by an action of a group uses a homomorphism into the automorphism group of the C^*-algebra. We know that, we can not talk about a homomorphism between an inverse semigroup and a group (automorphism group). The idea of a partial action is to replace the automorphism group by the inverse semigroup of partial automorphisms. By using the above facts, the definition of action of an inverse semigroup on a C^*-algebra is given by N. Sieben [13]. Also, invariant ideals of a partial action of a group, quotient partial actions and topological freeness for group actions are considered in [2]. Following [2,13], we are going to replace the group G by a unital inverse semigroup S and to discuss the above items.

The structure of the paper is as follows.

Action of an inverse semigroup on a C^*-algebra and its properties are considered in section 1. In section 2 we study topological actions of an inverse semigroup on a C^*-algebra. Also, we prove that for every $s\neq e$ in S there exists h in $C_0(X)$ such that $0\leq h\leq 1$ and $\|h(f\delta_s)h\|$ is too small for every f in $C_0(U_s)$.

Section 3 is devoted to consider the problem of invariant ideals, their associated quotient actions and the relation between ideals of A and ideals of $A\times_\alpha S$. Also, we prove that if $c\in C_0(X)\times_\alpha S$, ε is given and E is the conditional expectation on $C_0(X)\times_\alpha S$ then there exists $h\in C_0(X)$ such that $0\leq h\leq 1$ and $\|hE(c)h-hch\|<\varepsilon$.

ACTION OF AN INVERSE SEMIGROUP

By a *unital inverse semigroup* we mean a semigroup S with the unit element e such that for each s in S, there exists a unique element s^* in S with the following properties:

(i) $ss^*s = s$;
(ii) $s^*ss^* = s^*$.

Let A be a C^*-algebra. A *partial automorphism* of A is a triple (α,I,J) where I and J are closed two-sided ideals in A and $\alpha:I\to J$ is a $*$-isomorphism.

For given partial automorphisms (α,I,J) and (β,K,L) of A, their product $\alpha\beta$ is nothing but the composition of α and β with the largest possible domain, that is, $\alpha\beta:\beta^{-1}(I)\to A$ such that $(\alpha\beta)(a) = \alpha(\beta(a))$. Obviously, $\beta^{-1}(I)$ is a closed ideal of K and since ideals of ideals of a C^*-algebra are, themselves, ideals of that algebra, the product $(\alpha\beta, \beta^{-1}(I), \alpha\beta(\beta^{-1}(I)))$ is a partial automorphism too. It is not hard to see that the set PAut(A) of partial

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automorphisms of Ais a unital inverse semigroup under the composition with the largest possible domain with the identity (i,A,A), where i is the identity map on A and $(\alpha,I,J)^* = (\alpha^{-1},J,I)$.

Definition 1.1: Let A be a C^*-algebra and S be a unital inverse semigroup with the identity e. An *action* of S on A is a semigroup homomorphism. $s \mapsto (\alpha_s, E_{s^*}, E_s) : S \to PAut(A)$ with $E_e = A$.

An element s of an inverse semigroup S is called idempotent if $s^2 = s$. And S is called an idempotent semigroup if $s^2 = s$ for all s in S. Our general reference on semigroups is [7].

Lemma 1.2: Let S be an inverse semigroup, α an action of S on a C^*-algebra A and $s \in S$, then $\alpha_{s^*} = \alpha_s^{-1}, \alpha_e$ is the identity map on A and if s is an idempotent, then α_s is the identity map on $E_{s^*} = E_s$.

Proof: Since α is a homomorphism, we have

$\alpha_s = \alpha(s) = \alpha(ss^*s) = \alpha(s)\alpha(s^*)\alpha(s) = \alpha_s \alpha_{s^*} \alpha_s$

on the other hand $\alpha_s = \alpha_s \alpha_s^{-1} \alpha_s$ So, by the uniqueness of inverses in inverse semigroups, we conclude that $\alpha_{s^*} = \alpha_s^{-1}$.

Moreover

$$\alpha_e \alpha_s = \alpha_{es} = \alpha_s = \alpha_{se} = \alpha_s \alpha_e$$

therefore $\alpha_e = i_A$. If s is an idempotent, since $s^2 = s$ we have $sss = s^2 = s$ and $ss^*s = s$ so by uniqueness of inverse of s we conclude that $s = s^*$ and $\alpha_s = \alpha_{s^*}$. On the other hand

$$(\alpha_s)^2 = \alpha_s \alpha_s = \alpha_s \alpha_{s^*} = \alpha_e = i$$

Therefore α_s is the identity map on $E_s = E_{s^*}$.

Lemma 1.3: If α is an action of the unital inverse semigroup S on A, then $\alpha_t (E_{t^*} E_s) = E_{ts}$ for all s,t in S.

Proof: Since E_{t^*} and E_s are ideals in the C^*-algebra A we have $E_{t^*} E_s = E_{t^*} \cap E_s$. So

$$\begin{aligned} \alpha_t(E_{t^*} E_s) &= \alpha_t(E_{t^*} \cap E_s) = image\ (\alpha_t \alpha_s) \\ &= image(\alpha(t)\alpha(s)) \\ &= image(\alpha(ts)) \\ &= image(\alpha_{ts}) = E_{ts}. \end{aligned}$$

Definition 1.4: By a *semipartial dynamical system* we mean a triple (A,S,α), in which A is a C^*-algebra, S is a unital inverse semigroup and α is an action of S on A. The following definition is pivotal for our purposes.

Definition 1.5: Let (A,S,α) be semipartial dynamical system. By *a covariant representation* of (A,S,α) we mean a triple (π,ν,H) in which, $\pi: A \to B(H)$ is a non-degenerate representation of A on the Hilbert space H and $\nu: S \to B(H)$ is a multiplicative map such that

(i) $\nu_s \pi(a) \nu_{s^*} = \pi(\alpha_s(a))$ for all $a \in E_{s^*}$;

(ii) ν_s is a partial isometry with initial space $\pi(E_{s^*})H$ and final space $\pi(E_s)H$.

It is not hard to show that $\nu_e = 1_H$, the identity map on H and $\nu_{s^*} = (\nu_s)^*$.

Let (A,S,α) be a semipartial dynamical system. Consider

$$L_A = \{x \in \ell^1(S,A) : x(s) \in E_s\},$$

the closed subspace of $\ell^1(S,A)$. Define multiplication and involution on L_A by

$$(x * y)(s) = \sum_{rt=s} \alpha_r[\alpha_{r^*}(x(r))y(t)]$$

and,

$$x^*(s) = \alpha_s[x(s^*)^*]$$

Note that L_A is closed with respect to the above operations, simply because by the Lemma 1.3 we see that $(x*y)(s) \in E_s$ for every $s \in S$ and as a consequence $x*y \in L_A$. Also, for given x in L_A since $x(s^*) \in E_{s^*}$ and E_{s^*} is an ideal of A we have $(x(s^*))^* \in E_{s^*}$. Therefore $\alpha_s(x(s^*)^*) \in E_s$, that is $x^* \in L_A$. Simple computations show that $\|x*y\| \le \|x\|\|y\|$ and $\|x^*\| = \|x\|$ where $\|\cdot\|$ denotes the norm of L_A inherited from $\ell^1(S,A)$.

The fact that L_A is a Banach $*$-algebra is proved in [13, Prop. 4.1]. If (π,ν,H) is a covariant representation of (A,S,α), as defined by N. Sieben in [13], section 3, then $\pi \times \nu$ is a non-degenerate representation of L_A ([13, Prop. 4.3]).

Definition 1.6: Let A be a C^*-algebra and α be an action of the unital inverse semigroup S on A. Define a seminorm $\|\cdot\|_c$ on L_A by $\|x\|_c = \sup\{\|(\pi \times \nu)(x)\| : (\pi,\nu,H)$ is a covariant representation of $(A,S,\alpha)\}$. Let

$$I = \{x \in L_A : \| x \|_c = 0\}$$

The *crossed product* $A \times_\alpha S$ is the C^*-algebra obtained by completing the quotient L_A/I with respect to $\|\cdot\|_c$.

Example: The set of all pairs with complex coordinate, C^2, is a C^*-algebra with norm, multiplication and involution defined as follows:

$$\|(c_1, c_2)\| = \max\{|c_1|, |c_2|\};$$
$$(c_1, c_2)(c_1', c_2') = (c_1c_1', c_2c_2');$$
$$(c_1, c_2)^* = (\overline{c}_1, \overline{c}_2).$$

The group of integers, Z, is a unital inverse semigroup.

Take A = C^2 and S = Z. Define ideals E_0 = A,

$$E_{-1} = \{(a,0) : a \in A\}$$
$$E_1 = \{(0,a) : a \in A\}$$

and $E_n = \{(0,0)\}$ for all n, except for n = -1,0,1. Let α_0 be the identity map on A. Also, $\alpha_1((a,0)) = (0,a)$ is the forward shift and $\alpha_n = \alpha_1^n$ for all n≠0. Obviously, (α_n, E_{-n}, E_n) is a partial automorphism of A and $A\times_\alpha S$ is isomorphic to the matrix algebra M_2.

TOPOLOGICALLY FREE ACTIONS OF A UNITAL INVERSE SEMIGROUP

Topologically free partial actions of groups are considered in section 2 of [2]. Following that we will introduce the notion of *topological action* of an inverse semigroup on a locally compact Hausdorff space. Also we will consider those actions of an inverse semigroup on the C^*-algebra $C_0(X)$ which are corresponding to the above topological action.

The major new results of this section are theorems 2.4, 2.6 and 2.9.

In this section we will mostly be concerned with $(C_0(X),A,\alpha)$ where X is a locally compact Hausdorff space and α is that action of S on $C_0(X)$ which arises from partial homeomorphisms of X. That is, for every $s\in S$ there is an open subset U_s of X and a homeomorphism $\theta_s:U_{s^*}\to U_s$ such that $U_e = X$ and θ_e is the identity map on X. The action α of S on $C_0(X)$ corresponding to the partial homeomorphism θ is given by

$$\alpha_s(f)(x) = f(\theta_{s^*}(x))$$

for $s\in S$ and $f\in C_0(U_{s^*})$.

Now we can summarize the above facts in the following definition.

Definition 2.1: Let S be a unital inverse semigroup and X be a locally compact Hausdorff space a *topological action of S on X* is a pair $\theta = (\{U_s\}_{s\in S}, \{\theta_s\}_{s\in S})$, where for each s in S, U_s is an open subset of X,$\theta_s:U_{s^*}\to U_s$ is a homeomorphism, U_e = X and θ_e is the identity map on X.

Given a topological action $(\{U_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ of S on a locally compact Hausdorff space X, let $E_s = C_0(U_s)$ be identified, in the usual way, with the ideal of functions in $C_0(X)$ vanishing off U_s. Therefore we have the following definition.

Definition 2.2. The action α of S on $C_0(X)$ *corresponding to the topological action* θ is given by

$$\alpha_s(f)(x) := f(\theta_{s^*}(x)), f \in C_0(U_{s^*})$$

for each s in S.

Definition 2.3: The topological action θ of S on X is *topologically free* if for every $s\in S$-{e} the set

$$F_s := \{x \in U_{s^*} : \theta_s(x) = x\}$$

has empty interior.

Although F_s need not be closed in X, we will show that it is closed in U_{s^*}. For this, let x be a limit point of F_s and $x\in U_{s^*}$. There exists a net $\{x_i\}_i$ of elements of F_s such that $x_i\to x$. Since θ_s is a homeomorphism we have $\theta_s(x_i)\to\theta_s(x)$. From $\theta_s(x_i) = (x_i)$ we see that $x_i\to\theta_s(x)$. Uniqueness of the limit of a net shows that $\theta_s(x) = x$, that is, $x\in F_s$. This shows that F_s is closed in the domain of θ_s.

A set A in a topological space X is called *nowhere dense* if its closure has empty interior, in particular a closed set is nowhere dense if and only if its interior is empty. To say that A has empty interior is to say that A contains no open subset of X other than the empty set. The union of any finite set of nowhere dense sets is nowhere dense ([6], Sec. 1.10).

Theorem 2.4: The topological action θ of a unital inverse semigroup S on X is topologically free if and only if for every $s\in S$-{e}, the set F_s is nowhere dense.

Proof: The "if " part is trivial. For the "only if " let θ be topologically free. We know that F_s is closed relative to U_{s^*}. As a consequence $F_s = C\cap U_{s^*}$ in which C is a closed subset of X. If V is open and $V \subset \overline{F_s}$, then

$$V \cap U_{s^*} \subset \overline{F}_s \cap U_{s^*} = \overline{(C \cap U_{s^*})} \cap U_{s^*}$$
$$\subseteq \overline{C} \cap U_{s^*} = C \cap U_{s^*} = F_s.$$

Since F_s has empty interior and $V\cap U_{s^*}$ is open we see that $V\cap U_{s^*} = \varnothing$. So the open sets U_{s^*} and V are separated. Now, since

$$V \subset \overline{F}_s = \overline{C \cap U_{s^*}} \subseteq C \cap \overline{U_{s^*}} \subseteq \overline{U_{s^*}}$$

we see that V = ∅. That is F_s is nowhere dense.

The following equivalent version of topological freeness is more appropriate for our purposes.

Corollary 2.5: The topological action θ of S on X is topologically free if and only if for every finite subset $\{s_1, s_2,\dots,s_n\}$ of S-{e}, the set $\bigcup_{i=1}^{n} F_{s_i}$ has empty interior.

Proof: Any finite union of nowhere dense sets is nowhere dense.

In the remainder of this work we denote by $\delta_s (s\in S)$ the function in L which takes the value 1 at s and zero at every other element of S.

Theorem 2.6: Let $s \in S-\{e\}, f \in E_s = C_0(U_s)$ and $x_0 \notin F_s$. For every $\varepsilon>0$ there exists $h\in C_0(X)$ such that:

(i) $h(x_0)=1$;

(ii) $\| h(f\delta_s)h \| \le \varepsilon$ and

(iii) $0 \le h \le 1$.

Proof: Since $x_0 \notin F_s$ let us separate the proof into two cases according to x_0 being in the domain U_s of θ_{s^*} or not. Let $x_0 \notin U_s$. From $f\in E_s$ we see that the set

$$K := \{x \in U_s : |f(x)| \ge \varepsilon\}$$

is a closed subset of U_s and $x_0\notin K$. So by the Urysohn's lemma there exists h in $C_0(X)$ such that $0\le h\le 1, h(K) = 0$ and $h(x_0) = 1$.

Now since the restriction of the function h to the set U_s implies that $hf\in E_s$, we conclude that $(hf)\delta_s\in C_0(X)\times_\alpha S$. So that

$$\begin{aligned}\|((hf)\delta_s)(h\delta_e)\| \le \|hf\| &= \sup\{|h(x)f(x)| : x \in U_s\}\\ &= \sup(\{|h(x)f(x)| : x \in K\}\\ &\quad \cup \{|h(x)f(x)| : x \in U_s - K\})\\ &\le \varepsilon\end{aligned}$$

This shows that (ii) holds.

If $x_0\in U_s$ then $\theta_{s^*}(x_0)\neq x_0$ since X is Hausdorff, there are disjoint open sets V_1 and V_2 such that $x_0\in V_1\subset U_s$ and $\theta_{s^*}(x_0)\in V_2\subset U_{s^*}$.

If $V := \theta_s(V_2)\cap V_1$, then $x_0\in V_1$ and $\theta_{s^*}(V)\subset V_2$. Since $V_1\cap V_2 = \varnothing$ we have $\theta_{s^*}(V)\cap V = \varnothing$. Now there exists h in $C_0(X)$ such that $0\le h\le 1$, $h(x_0) = 1$ and $h(X-V) = 0$. Obviously, (i) and (iii) hold. To show (ii) holds, we know that

$$hf\delta_s h = ((hf)\delta_s)(h\delta_e) = \alpha_s(\alpha_{s^*}(hf)h)\delta_{se} = 0$$

simply because the support of $\alpha_{s^*}(hf)$ is contained in $\theta_{s^*}(V)$, the support of h is in V and $\theta_{s^*}(V)\cap V=\varnothing$.

Definition 2.7: If A is a C^*-algebra and if B is a C^*-subalgebra of A, then by a *conditional expectation* from A to B we mean a continuous positive projection of A onto B which satisfies the conditional expectation property

$$P(ba) = b\,P(a) \text{ and } P(ab) = P(a)\,b$$

for $b\in B$ and $a\in A$.

Note that if P is a positive map on A, then it is easily seen that $P(a^*) = (P(a))^*$ for all $a\in A$ and from this it follows easily that the conditional expectation property from right multiplication by elements of B is a consequence of that for left multiplication and conversely ([12, Def. 1.3.]).

By [11, 6.2.1] we can consider $C_0(X)$ as C^*-subalgebra of the partial crossed product $C_0(X)\times_\alpha S$. Therefore the conditional expectation from $C_0(X)\times_\alpha S$ onto $C_0(X)$ which is denoted by E is meaningful.

Definition 2.8: A semipartial dynamical system (A,S,α) is said to be *topologically free* if the set of fixed points for the partial homeomorphism associated to each non-trivial semigroup element has empty interior.

Since the conditional expectation $E: C_0(X)\times_\alpha S\to C_0(X)$ is contractive we can state and prove the following theorem.

Theorem 2.9: If $(C_0(X), S, \alpha)$ is a topologically free semipartial dynamical system then for every $c\in C_0(X)\times_\alpha S$ and every $\varepsilon>0$ there exists $h\in C_0(X)$ such that:

(i) $\|hE(c)h\| \ge \|E(c)\| - \varepsilon$,

(ii) $\|hE(c)h - hch\| \le \varepsilon$ and

(iii) $0 \le h \le 1$.

Proof: Let c be a finite linear combination of the form $\Sigma_{t\in T}\, \alpha_t\, \delta_t$, where T denotes a finite subset of S. Define $E(c) = a_e$ if $e\in T$ and $E(c) = 0$ if $e\notin T$. Since

$$\|a_e\| = \sup\{|a_e(x)| : x \in X\},$$

for given $\varepsilon>0$, the set

$$V = \{x \in X : |a_e(x)| \ge \|a_e\| - \varepsilon\}$$

which is, clearly open, is nonempty. Since the topological action α is topologically free by Corollary 2.5 there exists $x_0\in V$ such that $x_0\notin F_t$ for every $t\in T$. Take $f_t = a_t\delta_t\in D_t$, for $\varepsilon/|T|$ by Theorem 2.6 there exist functions h_t such that

$$h_t(x_0)=1,\ \|h_t(a_t\delta_t)h_t\| \le \frac{\varepsilon}{|T|} \text{ and } 0 \le h_t \le 1.$$

Let $h = \prod_{t\in T-\{e\}} h_t$. Obviously $0 \le \|h\| \le 1$, that is, (iii) holds. Also (i) holds, simply because $x_0 \in V$ and

$$\begin{aligned} \|ha_e h\| &= \sup\{|h(x)a_e(x)h(x)| : x \in X\} \\ &\ge |h(x_0)a_e(x_0)h(x_0)| \\ &= |a_e(x_0)| > \|a_e\| - \varepsilon \end{aligned}$$

In order to prove (ii), we have

$$\begin{aligned} \|ha_e h - hch\| &= \|ha_e h - \textstyle\sum_{t\in T} ha_t\delta_t h\| \\ &= \|\textstyle\sum_{t\in T-\{e\}} ha_t\delta_t h\| \\ &\le \textstyle\sum_{t\in T-\{e\}} \|ha_t\delta_t h\| \\ &< |T|\frac{\varepsilon}{|T|} = \varepsilon \end{aligned}$$

For arbitrary element c, since c is the limit of a net in $C_0(X)\times_\alpha S$ and E is contractive, a standard approximation argument finishes the proof.

PROPERTIES OF INVARIANT IDEALS

Throughout this section, S is a unital inverse semigroup, X is a locally compact Hausdorff space, θ is a topological action of S on X and α is the action of S on $C_0(X)$ which is corresponding to θ.

The major new results of this section are Lemma 3.3, Corollary 3.4, Theorem 3.5 and Conjecture 3.6.

Definition 3.1: An ideal I in $C_0(X)$ is Called *invariant* under the corresponding action α on $C_0(X)$ or simply *α-invariant* if $\alpha_s(I\cap E_{s^*}) \subseteq I$ for every s in S.

Lemma 3.2: If α is an action of S on a C^*-algebra $A = C_0(X)$ and I is an α-invariant ideal of A then

$$\alpha_t(E_{t^*}\cap I) = E_t \cap I\ .$$

Proof: Obviously, $\alpha_t(E_{s^*}\cap I) \subseteq E_t \cap I$. Now let $y \in E_t\cap I$. Since $y\in E_t$, there exists x in E_{t^*} such that $y = \alpha_t(x)$. We claim that $x\in I$ and as a consequence

$$y = \alpha_t(x) \in \alpha_t(E_{t^*}\cap I)$$

If $x\notin I$ then $x \notin E_{s^*}\cap I$ and $y = \alpha_t(x) \notin \alpha_t(E_{t^*}\cap I) \subset I$

That is, $y \notin I$ and it contradicts to the hypothesis.

Let α be an action of S on $A = C_0(X)$. For each invariant ideal I of A there is a restriction of α to an action of S on I. That is, if $\alpha = \{(\alpha_t, E_{t^*}, E_t)\}_{t\in S}$ is an action of S on A and $\alpha_t : E_{t^*} \to E_t$ is a partial automorphism of A, then

$$\Theta = \{(\theta_t, E_{t^*}\cap I, E_t\cap I)\}_{t\in S}$$

in which $\theta_t = \alpha_t|_I$ and

$$E_t\cap I = \theta_t(E_{t^*}\cap I)$$

is an action of S on I, by Lemma 3.2. Also,

$$\dot\alpha = \{(\dot\alpha_t, \dot E_{t^*}, \dot E_t)\}_{t\in S}$$

in which

$$\dot E_{t^*} = \{a + I \in A/I : a\in E_{t^*}\}$$

and,

$$\dot\alpha_t : \dot E_{t^*} \to \dot E_t = \alpha_t(E_{t^*}) + I$$

defined by

$$\dot\alpha_t(a+I) = \alpha_t(a) + I$$

is a *quotient action modulo I* of S on A/I.

Now we make an attempt to investigate the relation between the quotient of the crossed product $A\times_\alpha S$ modulo the ideal generated by I and the Crossed product of A/I by the quotient action modulo I. That is, the relation between $\frac{A\times_\alpha S}{\langle I\rangle}$ and $A/I \times_{\dot\alpha} S$.

Lemma 3.3: Let α be an action of S on a C^*-algebra A and I be an α-invariant ideal of A, then the map from $\ell^1(S,I)$ to $\ell^1(S,A)$ induces an injection from $I\times_\alpha S$ to $A\times_\alpha S$.

Proof: Let

$$L_A = \{x\in \ell^1(S,A) : x(s)\in E_s\}$$

and,

$$L_I = \{x\in \ell^1(S,I) : x(s)\in E_s\}$$

where in L_A the ideal E_s is an ideal of A but in L_I, the ideal E_s is an ideal of I. As we showed in [14], L_A and L_I are closed subalgebra of $\ell^1(S,A)$. The inclusion map from $\ell^1(S,I)$ into $\ell^1(S,A)$ maps L_I into L_A simply because if $b\in\ell^1(S,I)$, i.e., $b = \sum_{s\in S} a_s\delta_s$ where each $a_s\in E_s$, then $i(b) = b\in \ell^1(S,A)$. Note that we used the fact that, ideals of ideals of a C^*-algebra are, themselves, ideals of that algebra. Thus the inclusion map induces inclusion map i from $I\times_\alpha S$ to $A\times_\alpha S$. In order to prove that i is injective it is enough to show that every covariant representation of (I,S,α) extends to a covariant representation of (A,S,α). Therefore, let (π,ν,H) be an arbitrary covariant representation of (I,S,α). Since (π,H) is a representation of I without loss of generality we can assume that π:I$\rightarrow$B(H) is nondegenerate. By using [1, Prop. 2.10.4] there exists a unique extension π' of π to a representation of A on H and we have $v_s\pi'(a)v_{s^*} = \pi'(\alpha_s(a))$ for all a$\in E_{s^*}$. That is, (π',ν,H) is a covariant representation of (A,S,α).

Corollary 3.4: If I is an α-invariant closed two-sided ideal of A then I$\times_\alpha$S is a closed proper two-sided ideal of A$\times_\alpha$S.

We will denote by <J> the ideal generated by a subset J of a C^*-algebra B.

Theorem 3.5: Suppose α is an action of S on A and assume I is an α-invariant ideal of A. Then the map

$$a\delta_s \in I \times_\alpha S \rightarrow a\delta_s \in A \times_\alpha S$$

extends to an injection of I$\times_\alpha$S onto the ideal <I> generated by I in A$\times_\alpha$S and <I>$\cap$A = I.

Proof: Obviously, Lemma 3.3 and Corollary 3.4 show that I$\times_\alpha$S injects as an ideal in A$\times_\alpha$S. Therefore we can identify I$\times_\alpha$S with

$$\overline{sapn}\{a\delta_s : a \in E_s, s \in S\}.$$

Also, we can identify I with its canonical image Iδ_e in A$\times_\alpha$ S. Since <I> is the smallest ideal containing I we have <I>$\subseteq$I$\times_\alpha$S. In order to prove the reverse inclusion it suffices to show that a$\delta_s\in$<I> for every a$\in E_s\cap$I and s$\in$S. Therefore let a$\in E_s\cap$I and let b_λ be an approximate unit for the ideal E_s. Since

$$ab_\lambda\delta_s = (a\delta_e)(b_\lambda\delta_s) \in \langle I \rangle$$

and,

$$a\delta_\lambda = \lim_{\lambda\to\infty} ab_\lambda\delta_s \in \langle I \rangle$$

we have I$\times_\alpha$S$\subset$<I>. That is, I$\times_\alpha$S=<I> and as a consequence I = <I>$\cap$A.

Since the map a$\delta_s\rightarrow$(a+I)δ_s induces a $*$-homomorphism from ℓ^1(S,A) onto ℓ^1(S,A/I) we have the following conjecture.

Conjecture 3.6: Under the assumptions of Theorem 3.5 we have the following exact sequence.

$$0 \rightarrow I \times_\alpha S \rightarrow A \times_\alpha S \rightarrow (A/I) \times_{\dot{\alpha}} S \rightarrow 0.$$

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