

## Application of Optimal Homotopy Asymptotic Method to Special Sixth Order Boundary Value Problems

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**Abstract:** In this paper, the Optimal Homotopy Asymptotic Method (OHAM) is applied to derive solution of the special sixth order boundary value problem (BVP). It is interesting because it has got the involvement of constant  $c$ . Much better results are computed at various values of  $c$ . Furthermore, we can easily adjust the convergence domain and control the convergence region. As a result it is concluded that OHAM shows fast convergence, simple applicability and efficiency of the new technique.

**Key words:** Optimal Homotopy Asymptotic Method • Special Boundary Value Problem • Convergence Constants

### INTRODUCTION

Recently, Homotopy Perturbation Method (HPM) [1, 2], Adomian Decomposition Method (ADM) [1,2] and Differential Transform Method (DTM) [1,2] were applied to the special fourth and sixth order boundary value problem:

$$\begin{aligned} y^{(6)}(x) &= (1+c)y^{(4)}(x) \\ &- cy^{(2)}(x) + cx, \quad x \in [0, 1], \end{aligned} \quad (1)$$

With Boundary Conditions:

$$\begin{aligned} y(0) &= 1, \quad y(1) = \frac{7}{6} + \sinh(1), \\ y^{(1)}(0) &= 1, \quad y^{(1)}(1) = 1 + \cosh(1), \\ y^{(2)}(0) &= 0, \quad y^{(2)}(1) = 1 + \sinh(1). \end{aligned} \quad (2)$$

The Exact Solution of this Problem Is:

$$y(x) = 1 + \frac{1}{6}x^3 + \sinh(x) \quad (3)$$

Special BVP can be primarily modeled in [8]. In this paper, we apply new analytic technique OHAM, introduced very recently by Vasile Marinca *et al.* [3-7],

which is further development of the homotopy perturbation method [9-13], to solve special sixth order boundary value problem. The paper is organized as follows, in Section 2, the basics mathematical theory is given. In Section 3 OHAM is applied to the solution of special sixth order boundary value problem.

### BASIC IDEA OF OHAM

We apply OHAM to the following differential equation [3-6, 14, 15]:

$$\begin{aligned} L(y(x)) + g(x) + N(y(x)) &= 0, \\ B\left(y, \frac{dy}{dx}\right) &= 0 \end{aligned} \quad (4)$$

here  $L$  is a linear operator,  $x$  denotes independent variable,  $y(x)$  is an unknown function,  $g(x)$  is a known function,  $N$  is a nonlinear operator and  $B$  is a boundary operator.

According to OHAM a deformation equation is constructed as follows:

$$\begin{aligned} (1-q)[L(\phi(x, q)) + g(x)] &= H(q)[L(\phi(x, q)) + g(x) + N(\phi(x, q))], \\ B\left(\phi(x, q), \frac{\partial \phi(x, q)}{\partial x}\right) &= 0 \end{aligned} \quad (5)$$

Where  $q \in [0, 1]$  is an embedding parameter,  $H(q)$  is a nonzero auxiliary function for  $q \neq 0$  and  $H(0) = 0$ ,  $\phi(x, q)$  is an unknown function. Obviously, when  $q = 0$  and  $q=1$ , equation (5) results  $\phi(x, 0) = y_0(x)$  and  $\phi(x, 1) = y(x)$  respectively.

Thus, as  $q$  varies from 0 to 1, the solution  $\phi(x, q)$  varies from  $y_0(x)$  to the solution  $y(x)$ , where  $y(x)$  is obtained from Eq (7) for  $q = 0$ :

$$\begin{aligned} L(y_0(x)) + g(x) &= 0, \\ B\left(y_0, \frac{dy_0}{dx}\right) &= 0 \end{aligned} \quad (6)$$

We choose auxiliary function  $H(q)$  in the form

$$H(q) = qC_1 + q^2C_2 + \dots \quad (7)$$

Where  $C_1, C_2, \dots$  are constants to be determined later.

For solution, expanding  $\phi(x, q, C_i)$  in Taylor's series about  $q$ , we obtain:

$$\begin{aligned} \phi(z, q, C_i) &= y_0(x) \\ &+ \sum_{k=1}^{\infty} y_k(x, C_1, C_2, \dots, C_k) q^k \end{aligned} \quad (8)$$

Now substituting Eq. (8) into Eq. (5) and equating the coefficient of like powers of  $q$ , we obtained the following linear equations.

Zeroth order problem is given by Eq. (6) and the first and second order problems are given by the Eqs. (9-11) respectively:

$$\begin{aligned} L(y_1(x)) + g(x) &= C_1 N_0(y_0(x)), \\ B\left(y_1, \frac{dy_1}{dx}\right) &= 0 \end{aligned} \quad (9)$$

$$\begin{aligned} L(y_2(x)) - L(y_1(x)) &= C_2 N_0(y_0(x)) + \\ C_1 &\left[ L(y_1(x)) + N_1(y_0(x), y_1(x)) \right], \\ B\left(y_2, \frac{dy_2}{dx}\right) &= 0 \end{aligned} \quad (10)$$

The general governing equations for  $y_k(x)$  are given by

$$\begin{aligned} L(y_k(x)) - L(y_{k-1}(x)) &= C_k N_0(y_0(x)) \\ &+ \sum_{i=1}^{k-1} C_i \left[ \begin{aligned} &L(y_{k-i}(x)) \\ &+ N_{k-i}(y_0(x), y_1(x), \dots, y_{k-1}(x)) \end{aligned} \right], \\ k = 2, 3, \dots, \quad B\left(y_k, \frac{dy_k}{dx}\right) &= 0 \end{aligned} \quad (11)$$

Where  $N_m(y_0(x), y_1(x), \dots, y_{k-1}(x))$  is the coefficient of  $q^m$  in the expansion of  $N(\phi(x, q))$  about the embedding parameter  $q$ .

$$\begin{aligned} N(\phi(x, q, C_i)) &= N_0(y_0(x)) \\ &+ \sum_{m=1}^{\infty} N_m(y_0, y_1, \dots, y_m) q^m \end{aligned} \quad (12)$$

It has been observed that the Convergence of the series (8) depends upon the auxiliary constants  $C_1, C_2, \dots$ . If it is convergent at  $q = 1$ , one has

$$\begin{aligned} \tilde{y}(x, C_1, C_2, \dots, C_m) &= y_0(x) \\ &+ \sum_{i=1}^m y_i(x, C_1, C_2, \dots, C_m) \end{aligned} \quad (13)$$

Substituting Eq. (8) into Eq. (general problem), it results the following residual:

$$\begin{aligned} R(x, C_1, C_2, \dots, C_m) &= L(\tilde{y}(x, C_1, C_2, \dots, C_m)) \\ &+ g(x) + N(\tilde{y}(x, C_1, C_2, \dots, C_m)) \end{aligned} \quad (14)$$

If  $R = 0$ , then  $\tilde{y}$  will be the exact solution. Generally it doesn't happen, especially in nonlinear problems.

For the determinations of auxiliary constants  $C_i$ ,  $i = 1, 2, \dots, m$ , we choose  $a$  and  $b$  in a manner which leads to the optimum values of  $C_i$ ,  $s$  for the convergent solution of the desired problem. There are many methods like Galerkin's Method, Ritz Method, Collocation Method to find the optimal values of  $C_i$ ,  $i = 1, 2, 3, \dots$ . We apply the Method of Least Squares as under:

$$\begin{aligned} J(C_1, C_2, \dots, C_m) &= \int_a^b R^2(x, C_1, C_2, \dots, C_m) dx \end{aligned} \quad (15)$$

Where  $R$  is the residual,  $R = L(\tilde{y}) + g(x) + N(\tilde{y})$  and

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \dots = \frac{\partial J}{\partial C_m} = 0 \quad (16)$$

Where  $a$  and  $b$  are properly chosen numbers to locate the desired  $C_i (i = 1, 2, \dots, m)$ . With these constants known, the approximate solution (of order  $m$ ) is well-determined.

### APPLICATION OF OHAM

**Example:** In this example we apply OHAM to (1-2). Defining linear operator as:

$$L(\phi(x, q)) = \frac{\partial^6 \phi(x, q)}{\partial x^6}, \quad (17)$$

and choosing the nonlinear operator as:

$$N(\phi(x, q)) = (1 + c) \frac{\partial^4 \phi(x, q)}{\partial x^4} \quad (18)$$

$$-c \frac{\partial^2 \phi(x, q)}{\partial x^2}$$

and

$$g(x) = -c x \quad (19)$$

The Boundary Conditions Are:

$$\phi(0, q) = 1, \quad \phi(1, q) = \frac{7}{6} + \sinh(1), \quad (20)$$

$$\begin{aligned} \phi^{(1)}(0, q) &= 1, \quad \phi^{(1)}(1, q) = 1 + \cosh(1), \\ \phi^{(2)}(0, q) &= 0, \quad \phi^{(2)}(1, q) = 1 + \sinh(1). \end{aligned}$$

Equating coefficient of  $q^0, q^1, q^2, \dots$ , we get zeroth order, First order and second order problems as follows:

#### Zeroth Order Problem:

$$y_0^{(6)}(x) - c x = 0, \quad (21)$$

$$\begin{aligned} y_0(0) &= 1, \quad y_0(1) = \frac{7}{6} + \sinh(1), \\ y_0^{(1)}(0) &= 1, \quad y_0^{(1)}(1) = 1 + \cosh(1), \\ y_0^{(2)}(0) &= 0, \quad y_0^{(2)}(1) = 1 + \sinh(1). \end{aligned} \quad (22)$$

Its solution is

$$y_0(x) = \begin{cases} 1 + x + x^3 \left( -\frac{c}{1680} - 4 \text{Cosh}(1) + \frac{7}{6} (-5 + 9 \text{Sinh}(1)) \right) \\ + x^4 \left( 8 + \frac{c}{630} + 7 \text{Cosh}(1) - 16 \text{Sinh}(1) \right) \\ + x^5 \left( -3 - \frac{c}{840} - 3 \text{Cosh}(1) + \frac{13 \text{Sinh}(1)}{2} \right) + \frac{c}{5040} x^7 \end{cases} \quad (23)$$

#### First Order Problem:

$$\begin{aligned} y_1^{(6)}(x, C_1) - (1 + C_1) y_0^{(6)}(x) + c(1 + C_1) y_0^{(4)}(x) \\ - C_1 c^2 y_0^{(2)}(x) + c C_1 x + cx = 0 \end{aligned} \quad (24)$$

$$y_1(0) = y_1^{(1)}(0) = y_1^{(2)}(0) = y_1(1) = y_1^{(1)}(1) = y_1^{(2)}(1) = 0. \quad (25)$$

Its solution is

$$y_1(x, C_1) = \frac{(-1+x)^3 x^3}{279417600 e} \begin{cases} (-332640(37+44e-21e^2+5(1+e)(-19+7e)x)-22 \\ c(-15120(-37+95x)+e(665222+105840e(-3+5x) \\ +5x(-181470+7x(3+x))))+22c^2(e(-27662+5x \\ (3558+7(645-361x)x))+210e^2(80+x(-42+x \\ (-57+35x))-210(32+x(-150+x(-129+95x)))) \\ +c^3e(37+x(39+x(4+x(-68+7x(3+x)))))) \end{cases} C_1 \quad (26)$$

#### Second Order Problem:

$$\begin{aligned} y_2(x, C_1, C_2) + c x C_2 - c^2 C_2 y_0^{(2)}(x) - c^2 C_1 y_1^{(2)}(x, C_1) + C_2 y_0^{(4)}(x) + c C_2 y_0^{(6)}(x) \\ + C_1 y_1^{(4)}(x, C_1) + c C_1 y_1^{(4)}(x, C_1) - C_2 y_0^{(6)}(x) - y_1^{(6)}(x, C_1) - C_1 y_1^{(6)}(x, C_1) = 0, \end{aligned} \quad (27)$$

$$y_2(0) = y_2^{(1)}(0) = y_2^{(2)}(0) = y_2(1) = y_2^{(1)}(1) = y_2^{(2)}(1) = 0. \quad (28)$$

Its solution is

$$\begin{aligned}
 y_2(x, C_1, C_2) = & \frac{(-1+x)^3 x^3}{27461161728000e} \times \\
 & \left( 98280(c^3 e(37+39x+4x^2-68x^3+21x^4+7x^5)-332640(37+e(44-60x) \right. \\
 & -95x+7e^2(-3+5x))-22c(105840e^2(-3+5x)-15120(-37+95x)+ \\
 & e(665222-907350x+105x^2+35x^3))+22c^2(e(-27662+17790x+ \\
 & 22575x^2-12635x^3)+210e^2(80-42x-57x^2+35x^3)-210(32-150x \\
 & -129x^2+95x^3)))C_1+(c^5 e(965+1011x+123x^2-1699x^3-126x^4+630x^5 \\
 & +504x^6-504x^7+63x^8+21x^9)+12972960(-93526+239160x+4515x^2 \\
 & -3325x^3-12e(9286-12510x-315x^2+175x^3)+7e^2(7594-12540x-285x^2 \\
 & +175x^3))+2808c(64680e^2(3814-6240x-285x^2+175x^3)-9240(46906 \\
 & -119460x-4515x^2+3325x^3)+e(-517320906+688681610x+34846490x^2 \\
 & -19433330x^3+735x^4+245x^5))-2808c^2(105e^2(-134608+44488x+ \\
 & 179123x^2-107387x^3-2604x^4+980x^5)-105(-61840+221576x+ \\
 & 402371x^2-293459x^3-6188x^4+2660x^5)-2e(-12013446+4207190x \\
 & +17752070x^2-9668750x^3-252105x^4+88445x^5))-3c^3(98280e^2 \\
 & (-936-1712x+3563x^2+413x^3-2604x^4+980x^5)-98280(24+1136x \\
 & +5051x^2-859x^3-6188x^4+2660x^5)+e(164833286+275388546x \\
 & -615684729x^2-87422539x^3+471965592x^4-165616920x^5+8820x^6 \\
 & +2940x^7))+3c^4(210e^2(8408+6606x-3567x^2-22111x^3+19614x^4+ \\
 & 630x^5-5418x^6+1470x^7)-210(-5080-10350x-10551x^2-5683x^3+ \\
 & 31710x^4-3906x^5-13146x^6+3990x^7)-e(2678630+1899426x-1492689x^2 \\
 & -7576339x^3+7380912x^4+146160x^5-2072700x^6+532140x^7))) \\
 & +98280(c^3 e(37+39x+4x^2-68x^3+21x^4+7x^5)-332640(37+e(44-60x) \\
 & -95x+7e^2(-3+5x))-22c(105840e^2(-3+5x)-15120(-37+95x)+e(665222 \\
 & -907350x+105x^2+35x^3))+22c^2(e(-27662+17790x+22575x^2-12635x^3) \\
 & +210e^2(80-42x-57x^2+35x^3)-210(32-150x-129x^2+95x^3)))C_2
 \end{aligned}$$

(29)

Now Adding the above Equations We Have:

$$\tilde{y}(x, C_1, C_2) = y_0(x) + y_1(x, C_1) + y_2(x, C_1, C_2) \quad (30)$$

Substituting the approximate solution of the second order in equation (15), we get the residual R.

Now to find the optimal values of  $C_1, C_2$ , we will apply the method of Least Squares as mentioned above, we obtain as:

$$\begin{aligned}
 C_1 &= -0.9939356470997497, \\
 C_2 &= -0.00003879933613304636 \text{ for } c = 1
 \end{aligned}$$

Thus the above Equation Reduce to:

$$\tilde{y}(x) = \begin{cases} 1+x+0.3333333875851556x^3-2.6351381677525838 \times 10^{-9}x^4 \\ +0.00833321787998034x^5-4.583114876711522 \times 10^{-9}x^6 \\ +0.0001985723585094926x^7-1.5801598143928414 \times 10^{-7}x^8 \\ +2.8086542583364374 \times 10^{-6}x^9+1.7602798679708353 \times 10^{-9}x^{10} \\ +2.395089749788994 \times 10^{-8}x^{11}-3.3338633864656773 \times 10^{-12}x^{12} \\ -4.751453301896842 \times 10^{-10}x^{13}+7.554694767677805 \times 10^{-13}x^{15} \end{cases} \quad (31)$$

Table 1:

| x   | ADM                   | HPM                   | DTM                  | OHAM                      |
|-----|-----------------------|-----------------------|----------------------|---------------------------|
| 0.0 | 0.0                   | 0.0                   | 0.0                  | 0.0                       |
| 0.1 | $7.8 \times 10^{-10}$ | $7.8 \times 10^{-10}$ | $4.5 \times 10^{-6}$ | $5.01488 \times 10^{-12}$ |
| 0.2 | $4.7 \times 10^{-9}$  | $4.7 \times 10^{-9}$  | $2.5 \times 10^{-5}$ | $3.4224 \times 10^{-11}$  |
| 0.3 | $1.7 \times 10^{-8}$  | $1.7 \times 10^{-8}$  | $5.9 \times 10^{-5}$ | $8.92695 \times 10^{-11}$ |
| 0.4 | $1.9 \times 10^{-8}$  | $1.9 \times 10^{-8}$  | $9.1 \times 10^{-5}$ | $1.45841 \times 10^{-10}$ |
| 0.5 | $2.4 \times 10^{-8}$  | $2.4 \times 10^{-8}$  | $1.0 \times 10^{-4}$ | $1.7111 \times 10^{-10}$  |
| 0.6 | $2.3 \times 10^{-8}$  | $2.3 \times 10^{-8}$  | $9.6 \times 10^{-5}$ | $1.48808 \times 10^{-10}$ |
| 0.7 | $1.7 \times 10^{-8}$  | $1.7 \times 10^{-8}$  | $6.6 \times 10^{-5}$ | $9.24671 \times 10^{-11}$ |
| 0.8 | $8.6 \times 10^{-9}$  | $8.6 \times 10^{-9}$  | $3.0 \times 10^{-5}$ | $3.56235 \times 10^{-11}$ |
| 0.9 | $1.7 \times 10^{-9}$  | $1.7 \times 10^{-9}$  | $5.5 \times 10^{-6}$ | $5.17764 \times 10^{-12}$ |
| 1.  | 0.0                   | 0.0                   | 0.0                  | 0.0                       |

Table 2

| x   | ADM                  | HPM                  | DTM                  | OHAM                      |
|-----|----------------------|----------------------|----------------------|---------------------------|
| 0.0 | 0.0                  | 0.0                  | 0.0                  | 0                         |
| 0.1 | $1.2 \times 10^{-6}$ | $1.2 \times 10^{-6}$ | $2.9 \times 10^{-5}$ | $7.13252 \times 10^{-8}$  |
| 0.2 | $7.2 \times 10^{-6}$ | $7.2 \times 10^{-6}$ | $1.6 \times 10^{-4}$ | $3.78362 \times 10^{-7}$  |
| 0.3 | $1.7 \times 10^{-5}$ | $1.7 \times 10^{-5}$ | $3.6 \times 10^{-4}$ | $8.04295 \times 10^{-7}$  |
| 0.4 | $2.7 \times 10^{-5}$ | $2.7 \times 10^{-5}$ | $5.3 \times 10^{-4}$ | $1.12364 \times 10^{-6}$  |
| 0.5 | $3.4 \times 10^{-5}$ | $3.4 \times 10^{-5}$ | $6.0 \times 10^{-4}$ | $1.18255 \times 10^{-6}$  |
| 0.6 | $3.2 \times 10^{-5}$ | $3.2 \times 10^{-5}$ | $5.3 \times 10^{-4}$ | $9.68295 \times 10^{-7}$  |
| 0.7 | $2.3 \times 10^{-5}$ | $2.3 \times 10^{-5}$ | $3.5 \times 10^{-4}$ | $5.95893 \times 10^{-7}$  |
| 0.8 | $1.1 \times 10^{-5}$ | $1.1 \times 10^{-5}$ | $1.5 \times 10^{-4}$ | $2.39795 \times 10^{-7}$  |
| 0.9 | $2.2 \times 10^{-6}$ | $2.2 \times 10^{-6}$ | $2.7 \times 10^{-4}$ | $3.83349 \times 10^{-8}$  |
| 1.  | 0.0                  | 0.0                  | 0.0                  | $4.44089 \times 10^{-16}$ |

Table 3:

| x   | ADM                  | HPM                  | DTM                  | OHAM                      |
|-----|----------------------|----------------------|----------------------|---------------------------|
| 0.0 | 0.0                  | 0.0                  | 0.0                  | 0                         |
| 0.1 | $2.1 \times 10^{-5}$ | $2.1 \times 10^{-5}$ | $4.7 \times 10^{-5}$ | $2.0838 \times 10^{-7}$   |
| 0.2 | $1.4 \times 10^{-4}$ | $1.4 \times 10^{-4}$ | $3.2 \times 10^{-4}$ | $8.52908 \times 10^{-7}$  |
| 0.3 | $4.1 \times 10^{-4}$ | $4.1 \times 10^{-4}$ | $8.7 \times 10^{-4}$ | $1.21482 \times 10^{-6}$  |
| 0.4 | $7.5 \times 10^{-4}$ | $7.5 \times 10^{-4}$ | $1.5 \times 10^{-3}$ | $7.80319 \times 10^{-7}$  |
| 0.5 | $1.0 \times 10^{-3}$ | $1.0 \times 10^{-3}$ | $2.1 \times 10^{-3}$ | $2.32466 \times 10^{-7}$  |
| 0.6 | $1.1 \times 10^{-3}$ | $1.1 \times 10^{-3}$ | $2.2 \times 10^{-3}$ | $1.14008 \times 10^{-6}$  |
| 0.7 | $9.2 \times 10^{-3}$ | $9.2 \times 10^{-3}$ | $1.7 \times 10^{-3}$ | $1.35982 \times 10^{-6}$  |
| 0.8 | $4.9 \times 10^{-3}$ | $4.9 \times 10^{-3}$ | $9.1 \times 10^{-4}$ | $8.57547 \times 10^{-7}$  |
| 0.9 | $1.0 \times 10^{-4}$ | $1.0 \times 10^{-4}$ | $1.9 \times 10^{-4}$ | $1.98607 \times 10^{-7}$  |
| 1.  | 0.0                  | 0.0                  | 0.0                  | $8.88178 \times 10^{-16}$ |

Results can be seen in Table 1-4 for various values of  $c$  by OHAM and compare with HPM [2], ADM [2] and DTM [2]. We observed that OHAM solution is much better than HPM [2], ADM [2] and DTM [2]. The beauty of OHAM can be seen from the Table 5 which is constructed to check the convergence of OHAM. Table 5, shows the fast convergence of OHAM. Moreover unlike DTM, OHAM produced good results at extended domain.

Further, in Table 5, we have shown that convergence depends on the number of convergence constants  $C_1, C_2, \dots$ . In zeroth order problem we use no convergence constant, so we get accuracy up to  $10^{-7}$  range. While in the first order problem we introduced a constant  $C_1$  and we see from the Table 5 that the accuracy has been

Table 4:

| x   | ADM                  | HPM                  | DTM                   | OHAM                     |
|-----|----------------------|----------------------|-----------------------|--------------------------|
| 0.0 | 0.0                  | 0.0                  | 0.0                   | 0                        |
| 0.1 | $1.4 \times 10^{-3}$ | $1.4 \times 10^{-3}$ | $5.9 \times 10^{-5}$  | $2.07403 \times 10^{-6}$ |
| 0.2 | $1.0 \times 10^{-2}$ | $1.0 \times 10^{-2}$ | $4.0 \times 10^{-4}$  | $7.95168 \times 10^{-6}$ |
| 0.3 | $3.2 \times 10^{-2}$ | $3.2 \times 10^{-2}$ | $1.1 \times 10^{-3}$  | 0.0000100271             |
| 0.4 | $6.3 \times 10^{-2}$ | $6.3 \times 10^{-2}$ | $1.9 \times 10^{-3}$  | $4.21226 \times 10^{-6}$ |
| 0.5 | $9.3 \times 10^{-2}$ | $9.3 \times 10^{-2}$ | $2.6 \times 10^{-3}$  | $5.78429 \times 10^{-6}$ |
| 0.6 | $1.0 \times 10^{-1}$ | $1.0 \times 10^{-1}$ | $2.8 \times 10^{-3}$  | 0.0000128639             |
| 0.7 | $8.6 \times 10^{-2}$ | $8.6 \times 10^{-2}$ | $2.2 \times 10^{-3}$  | 0.000012749              |
| 0.8 | $4.7 \times 10^{-2}$ | $4.7 \times 10^{-2}$ | $1.1 \times 10^{-3}$  | $7.06115 \times 10^{-6}$ |
| 0.9 | $1.0 \times 10^{-2}$ | $1.0 \times 10^{-2}$ | $2.5 \times 10^{-4}$  | $1.46475 \times 10^{-6}$ |
| 1.  | 0.0                  | 0.0                  | $7.1 \times 10^{-10}$ | $1.77636 \times 10^{15}$ |

Table 5: Convergence of OHAM Solutions

| x   | Zeroth Order OHAM        | First Order OHAM          | Second Order OHAM         |
|-----|--------------------------|---------------------------|---------------------------|
| 0.0 | 0.0                      | 0.0                       | 0.0                       |
| 0.1 | $2.17744 \times 10^{-8}$ | $2.87424 \times 10^{-10}$ | $5.01488 \times 10^{-12}$ |
| 0.2 | $1.30402 \times 10^{-7}$ | $2.0029 \times 10^{-9}$   | $3.4224 \times 10^{-11}$  |
| 0.3 | $3.14964 \times 10^{-7}$ | $5.25813 \times 10^{-9}$  | $8.92695 \times 10^{-11}$ |
| 0.4 | $5.03289 \times 10^{-7}$ | $8.67476 \times 10^{-9}$  | $1.45841 \times 10^{-10}$ |
| 0.5 | $6.10095 \times 10^{-7}$ | $1.03838 \times 10^{-8}$  | $1.7111 \times 10^{-10}$  |
| 0.6 | $5.7986 \times 10^{-7}$  | $9.32644 \times 10^{-9}$  | $1.48808 \times 10^{-10}$ |
| 0.7 | $4.17906 \times 10^{-7}$ | $6.04601 \times 10^{-9}$  | $9.24671 \times 10^{-11}$ |
| 0.8 | $1.99081 \times 10^{-7}$ | $2.4308 \times 10^{-9}$   | $3.56235 \times 10^{-11}$ |
| 0.9 | $3.82003 \times 10^{-8}$ | $3.5721 \times 10^{-10}$  | $5.17764 \times 10^{-12}$ |
| 1.  | 0.0                      | 0.0                       | 0.0                       |

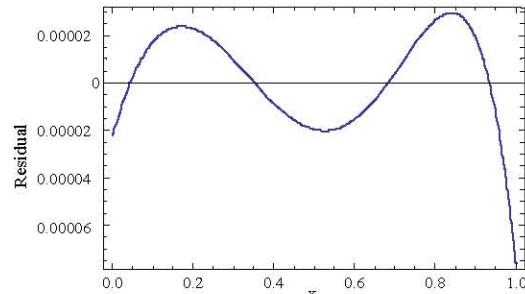


Fig. 1: The accuracy of second order OHAM can easily be seen from the plot for Residual.

increased. In the second order problem we introduce another constant  $C_2$  and hence the accuracy increases. Thus we can conclude that the accuracy is directly proportional to the number of convergence constants  $C_i$  involving in the auxiliary function  $H(q)$ .

## CONCLUSION

We applied a new powerful analytic method, OHAM to derive solution of the special sixth order boundary value problem (BVP). The BVPs involve a constant  $c$ , the increasing value  $c$  effect the results badly while solving it analytically by ADM [2], HPM[2]. It is seen from the Tables 1-5, that we get highly accurate results by even lower order approximation. This method

provides us, a convenient way to control the convergence and we can easily adjust the desired convergence regions. This approach is simple in applicability, as it does not require discretization or perturbation like other numerical and approximate methods. Moreover, this technique is fast converging to the exact solution and requires less computational work. This confirms our belief that the efficiency of the OHAM gives it much wider applicability.

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