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Application of Optimal Homotopy Asymptotic Method to Fourth Order Boundary Value Problems

[1]*M. Idrees,* [1]*Sirajul Haq and* [2]*S. Islam*

[1]Faculty of Engineering Sciences, GIK Institute, Topi 23640, NWFP, Pakistan
[2]Department of Mathematics, CIIT, Park Road Chakshazad, Islamabad, Pakistan

Abstract: In this paper, the Optimal Homotopy Asymptotic Method (OHAM) applied to derive solution of fourth order linear and nonlinear boundary value problems. It is observed that OHAM is independent of the free parameter and we get better accuracy. Moreover we can easily adjust the convergence domain and control the convergence region .As a result it is concluded that OHAM show fast convergence, simple applicability and efficiency of the new technique.

Key words: Optimal Homotopy Asymptotic Method · Convergence Constants

INTRODUCTION

Differential equations can be solved analytically by various perturbation techniques [1-3]. These techniques are very simple in calculating the solutions, but the limitations of these methods are based on the assumption of small parameter and there is no proper way of its selection. The researchers were looking for some new techniques which are independent of the small parameter.

In the last decade, the idea of homotopy was combined with perturbation. The fundamental work was done by S.J.Liao and Ji-Huan He. Liao proposed Homotopy Analysis Method (HAM) in his Ph.D dissertations [4]. This method involves a free parameter $\hbar$, whose suitable choice results into fast convergence. Ji-Huan He introduced Homotopy Perturbation Method (HPM) [4-10]. These methods are independent of the assumption of small parameter as well as they cover all the advantages of the perturbation method. The comprehensive comparison between HAM and HPM is studied in [7].

In this race of research very recently Vasile Marinca *et al*. introduced OHAM [13-16] for the solution of nonlinear problems of thin film flow of a fourth grade fluid down a vertical cylinder and for the study of the behavior of nonlinear mechanical vibration of electrical machine. The same author [13-16] used this method for the solution of nonlinear equations arising in the steady state flow of a fourth-grade fluid past a porous plate and for the solution of nonlinear equation arising in heat transfer. Moreover it is observed [13-16] that HPM and HAM are the special cases of OHAM.

In this paper we use OHAM to study the. The paper is organized as follows, in Section 2, the basics equations are derived. Section 4 is reserved for the basic mathematical theory of OHAM. In Section 5 OHAM is applied to the solution of given problem. In Section 6 the convergence constant are determined and discussed.

BASIC IDEA OF OHAM

We apply OHAM to the following differential equation [13-16]:

$$L(y(x)) + g(x) + N(y(x)) = 0, \quad B\left(y, \frac{dy}{dx}\right) = 0 \tag{1}$$

Where L is a linear operator, x denotes independent variable, $y(x)$ is an unknown function, $g(x)$ is a known function, N is a nonlinear operator and B is a boundary operator.

According to OHAM a deformation equation is constructed:

$$(1-p)[L(\phi(x,p) + g(x)] = H(p)[L(\phi(x,p) + g(x) + N(\phi(x,p))], \quad B\left(\phi(x,p), \frac{\partial\phi(x,p)}{\partial x}\right) = 0 \tag{2}$$

Corresponding Author: M. Idrees, Faculty of Engineering Sciences, GIK Institute, Topi 23640, NWFP, Pakistan

Where $p\in[0,1]$ is an embedding parameter, $H(p)$ is a nonzero auxiliary function for $p\neq 0$ and $H(0)=0$, $\phi(x,p)$ is an unknown function. Obviously, when $p=0$ and $p=1$ it holds $\phi(x,0) = y_0(x)$ and $\phi(x,0) = y(x)$ espectively.

Thus, as p varies from 0to1, the solution $\phi(x,p)$ varies from $y_0(x)$ to the solution $y(x)$, where $y_0(x)$ is obtained from Eq (2) for $p = 0$:

$$L\left(y_0(x)\right)+g(x)=0, \quad B\left(y_0,\frac{dy_0}{dx}=0\right) \tag{3}$$

We choose auxiliary function $H(p)$ in the form

$$h(p)=pC_1+p^2C_2+\ldots \tag{4}$$

Where $C_1C_2,\ldots$ are constants to be determined later. For solution, expanding $\phi(x, p, C_i)$ in Taylor's series about p, we obtain:

$$\phi(z,p,C_i)=y_0(x)+\sum_{k=1}^{\infty}y_k(x,C_1,C_2,\ldots,C_k)p^k \tag{5}$$

Now substituting Eq. (5) into Eq. (2) and equating the coefficient of like powers of p, we obtained the following linear equations.

Zeroth order problem is given by Eq. (3) and the first and second order problem are given by the Eqs. (6-7) respectively:

$$L\left(y_1(x)\right)+g(x)=C_1N_0\left(y_0(x)\right), \quad B\left(y_1,\frac{dy_1}{dx}\right)=0 \tag{6}$$

$$L\left(y_2(x)\right)-L\left(y_1(x)\right)=C_2N_0\left(y_0(x)\right)+C_1\left[L\left(y_1(x)\right)+N_1\left(y_0(x),y_1(x)\right)\right], \quad B\left(y_2,\frac{dy_2}{dx}\right)=0 \tag{7}$$

The general governing equations for $y_k(x)$ are given by

$$L\left(y_k(x)\right)-L\left(y_{k-1}(x)\right)=C_kN_0\left(y_0(x)\right)+\sum_{i=1}^{k-1}C_i\left[\begin{array}{l}L\left(y_{k-i}(x)\right)\\+N_{k-i}\left(y_0(x),y_1(x),\ldots,y_{k-1}(x)\right)\end{array}\right], \quad k=2,3,\ldots, \quad B\left(y_k,\frac{dy_k}{dx}\right)=0 \tag{8}$$

Where $N_m(y_0(x), y_1(x),\ldots, y_{k-1}(x))$ is the coefficient of p^m in the expansion of $N(\phi(x, p, C_i))$ about the embedding parameter p.

$$N\left(\phi(x,p,C_i)\right)=N_0\left(y_0(x)\right)+\sum_{m=1}^{\infty}N_m\left(y_0,y_1,y_2,\ldots,y_m\right)p^m \tag{9}$$

It has been observed that the Convergence of the series (5) depends upon the auxiliary constants C_1, C_2,.... If it is convergent at $p = 1$, one has

$$\tilde{y}(x,C_1,C_2,\ldots,C_m)=y_0(x)+\sum_{i=1}^{m}y_i(x,C_1,C_2,\ldots,C_m) \tag{10}$$

Substituting Eq. (10) into Eq. (general problem) it results the following residual:

$$R(x,C_1,C_2,\ldots,C_m)=L(\tilde{y}(x,C_1,C_2,\ldots,C_m))+g(x)+N(\tilde{y}(x,C_1,C_2,\ldots,C_m)) \tag{11}$$

If $R = 0$. then $\tilde{y}$ will be the exact solution. Generally it doesn't happen, especially in nonlinear problems.

For the determinations auxiliary constants of C_i, $I = 1,2,\ldots,m$, we choose a and b in a manner which leads to the optimum values of C_i, s for the convergent solution of the desired problem. There are many methods like Galerkin's Method, Ritz Method, Collocation Method to find the optimal values of C_i, $I = 1,2,3,\ldots$, We apply the Method of Least Squares as under:

$$J(C_1,C_2,\ldots,C_m)=\int_a^b R^2(x,C_1,C_2,\ldots,C_m)dx \tag{12}$$

Where R is the residual, $R=L(\tilde{y})+g(x)+N(\tilde{y})$ and

$$\frac{\partial J}{\partial C_1}=\frac{\partial J}{\partial C_2}=\ldots=\frac{\partial J}{\partial C_m}=0 \tag{13}$$

Where a and b are properly chosen numbers to locate the desired $C_i(i=1,2,\ldots m)$. With these constants known, the approximate solution (of order m) is well-determined.

APPLICATION OF OHAM

Example 1: Consider the following fourth order nonlinear boundary value problem [17]:

$$y^{(iv)}(x)+g(x)+\left(y''(x)\right)^2=0, 0\le x\le 1 \tag{14a}$$

$$y(0)=0,\ \ y(1)=\sin(1), \tag{14b}$$
$$y'(0)=1,\ \ y'(1)=\cos(1)$$

and $g(x)=-\sin(x)-\sin^2(x)$.

Zeroth Order Problem:

$$y_0^{(iv)}(x)+g(x)=0 \tag{15a}$$

$$y_0(0)=0, y_0(1)=\sin(1), \tag{15b}$$
$$y_0'(0)=1,\ y_0'(1)=\cos(1)$$

First Order Problem:

$$y_1^{(iv)}(x,C_1)=(1+C_1)y_0^{(iv)}(x) \tag{16a}$$
$$+(1+C_1)g(x)+C_1\left(y_0''(x)\right)^2$$

$$y_1(0)=y_1(1)=y_1'(0)=y_1'(1)=0 \tag{16b}$$

Second Order Problem:

$$y_2^{(iv)}(x,C_1,C_2)=(1+C_1)y_1^{(iv)}(x,C_1) \tag{17a}$$
$$+2C_1y_0''(x)y_1''(x,C_1)$$
$$+C_2y_0^{(iv)}(x)+C_2g(x)+C_2\left(y_0''(x)\right)^2$$

$$y_2(0)=y_2(1)=y_2'(0)=y_2'(1)=0 \tag{17b}$$

Solving problems (15a)-(17b) in succession we obtain the following the second order solution $\tilde{y}=y_0(x)+y_1(x,C_1)+y_2(x,C_1,C_2)$ C_1, C_2 can be easily calculated by the method of least squares as mention in the section 3. Result of example1 can be seen in Table 1 and Fig. 1.

Example 2: Consider the following linear problem [17]:

$$y^{(iv)}(x)=y^2(x)-x^{10}+4x^9-4x^8 \tag{18a}$$
$$-4x^7+8x^6-4x^4+120x-48$$

$$y(0)=y'(0)=0, \tag{18b}$$
$$y(1)=1,\ y'(1)=1$$

Table 1:

x	Exact	OHAM	Absolute Error
0.0	0.0	2.0961×10^{-13}	2.0961×10^{-13}
0.1	0.0998334	0.0998335	3.44091×10^{-8}
0.2	0.198669	0.198669	1.13845×10^{-7}
0.3	0.29552	0.29552	2.06689×10^{-7}
0.4	0.389418	0.389419	2.82662×10^{-7}
0.5	0.479426	0.479426	3.15173×10^{-7}
0.6	0.564642	0.564643	2.90029×10^{-7}
0.7	0.644218	0.644218	2.14158×10^{-7}
0.8	0.717356	0.717356	1.16136×10^{-7}
0.9	0.783327	0.783327	3.40555×10^{-8}
1.0	0.841471	0.841471	1.44107×10^{-13}

Table 2:

x	Exact	OHAM	Absolute Error
0.0	0.0	0.0	0.0
0.1	0.01981	0.019809	7.27908×10^{-8}
0.2	0.07712	0.0771198	2.45216×10^{-7}
0.3	0.16623	0.16623	4.48868×10^{-7}
0.4	0.27904	0.279039	6.18615×10^{-7}
0.5	0.40625	0.406249	6.99951×10^{-7}
0.6	0.53856	0.538559	6.6128×10^{-7}
0.7	0.66787	0.667869	5.07446×10^{-7}
0.8	0.78848	0.78848	2.87104×10^{-7}
0.9	0.89829	0.89829	8.56656×10^{-8}
1.0	1.0	1.0	0.0

Table 3:

x	Exact	OHAM	Absolute Error
-1	0	0.0	0.0
-0.9	0.0201954	0.0201954	2.42861×10^{-17}
-0.8	0.0397693	0.0397693	4.16334×10^{-17}
-0.7	0.0581837	0.0581837	4.85723×10^{-17}
-0.6	0.074985	0.074985	4.16334×10^{-17}
-0.5	0.0897962	0.0897962	2.77556×10^{-17}
-0.4	0.102311	0.102311	0.0
-0.3	0.112286	0.112286	4.16334×10^{-17}
-0.2	0.119538	0.119538	4.16334×10^{-17}
-0.1	0.12394	0.12394	4.16334×10^{-17}
0.0	0.125416	0.125416	0.0
0.1	0.12394	0.12394	1.38778×10^{-16}
0.2	0.119538	0.119538	8.32667×10^{-17}
0.3	0.112286	0.112286	2.77556×10^{-17}
0.4	0.102311	0.102311	8.32667×10^{-17}
0.5	0.0897962	0.0897962	0.0
0.6	0.074985	0.074985	9.71445×10^{-17}
0.7	0.0581837	0.0581837	1.38778×10^{-17}
0.8	0.0397693	0.0397693	7.63278×10^{-17}
0.9	0.0201954	0.0201954	1.07553×10^{-16}
.0	0.	3.96466E-17	3.96466×10^{-17}

Table 4:

x	Exact	OHAM (Zero Order)	Absolute Error
0.0	2.0	2.0	2.88658×10^{-15}
0.1	2.09982	2.09982	0.0
0.2	2.19852	2.19852	0.
0.3	2.29476	2.29476	2.66454×10^{-15}
0.4	2.38692	2.38692	0.
0.5	2.47308	2.47308	2.66454×10^{-15}
0.6	2.55097	2.55097	1.77636×10^{-15}
0.7	2.61788	2.61788	3.55271×10^{-15}
0.8	2.67065	2.67065	3.55271×10^{-15}
0.9	2.70556	2.70556	2.22045×10^{-15}
1.0	2.71828	2.71828	4.44089×10^{-15}
1.1	2.70375	2.70375	2.22045×10^{-15}
1.2	2.65609	2.65609	8.88178×10^{-16}
1.3	2.56851	2.56851	3.10862×10^{-15}
1.4	2.43312	2.43312	0.0
1.5	2.24084	2.24084	1.33227×10^{-15}
1.6	1.98121	1.98121	3.55271×10^{-15}
1.7	1.64218	1.64218	1.11022×10^{-15}
1.8	1.20993	1.20993	1.55431×10^{-15}
1.9	0.668589	0.668589	2.22045×10^{-16}
2.0	0.0	1.23214E-15	1.23214×10^{-15}

Table 5:

x	VIM	OHAM
0.0	0.	0.0
0.1	0.00150566	0.00151414
0.2	0.00540549	0.00543754
0.3	0.0108488	0.0109167
0.4	0.0170848	0.017198
0.5	0.023463	0.0236281
0.6	0.0294325	0.0296535
0.7	0.0345429	0.03482060.8
	0.0384436	0.03877620.9
	0.0408841	0.0412669
1.0	0.0417144	0.0421394
1.1	0.0408841	0.0413405
1.2	0.0384436	0.038917
1.3	0.0345429	0.0350157
1.4	0.0294325	0.0298836
1.5	0.023463	0.0238686
1.6	0.0170848	0.0174205
1.7	0.0108488	0.0110928
1.8	0.00540549	0.00554551
1.9	0.00150566	0.00155084
2.0	-8.56316E-19	4.85017E-13

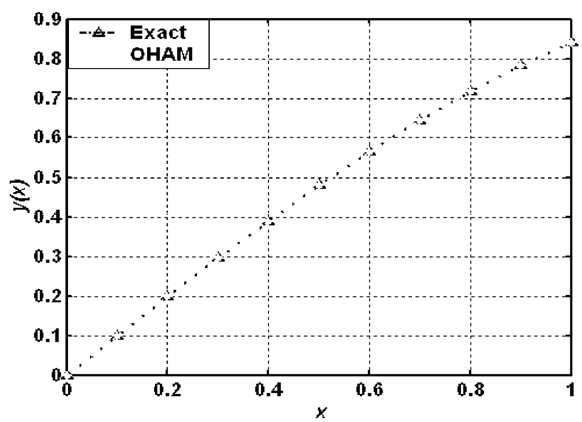

Fig. 1:

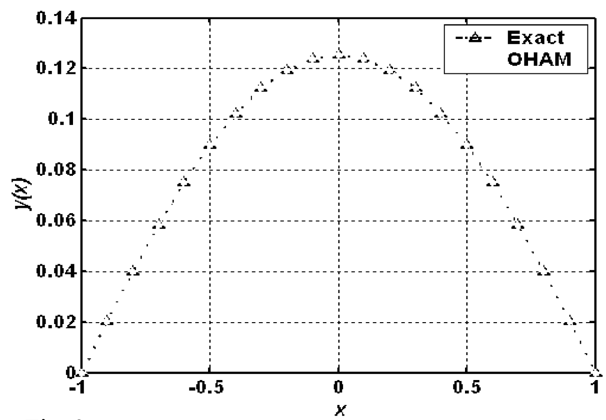

Fig. 3:

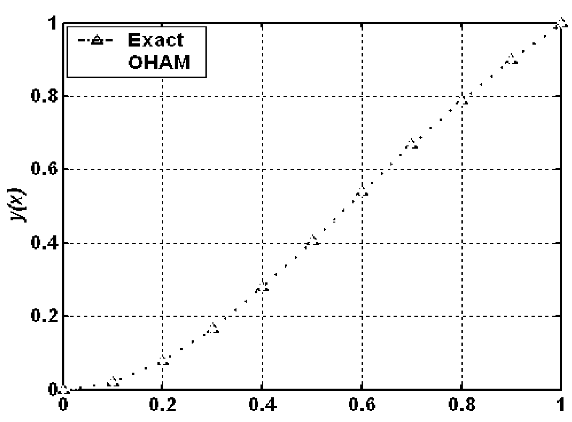

Fig. 2:

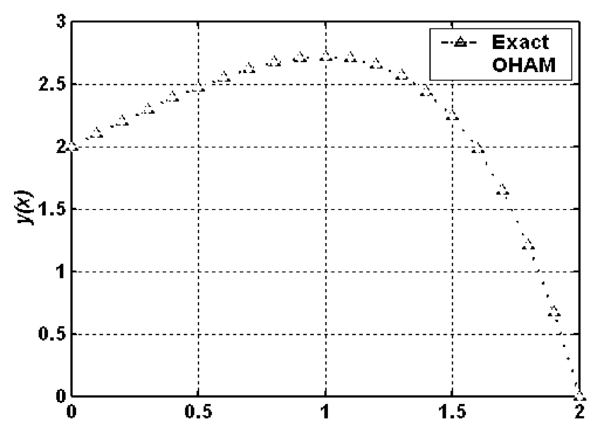

Fig. 4:

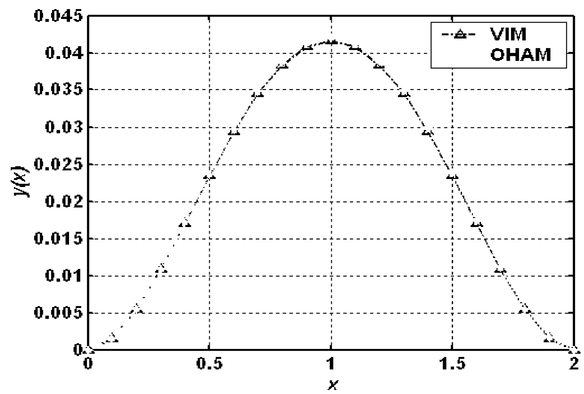

Fig. 5:

The Analytic Solution of this Problem Is:

$$y(x) = x^5 - 2x^4 + 2x^2. \tag{19}$$

Let

$$g(x) = x^{10} - 4x^9 + 4x^8 + 4x^7$$
$$-8x^6 + 4x^4 - 120x + 48$$

$$y^{(vi)}(x) - (y(x))^2 + g(x) = 0$$

Zeroth Order Problem:

$$y_0^{(iv)}(x) + g(x) = 0 \tag{20a}$$

$$y_0(0) = 0,\ y_0(1) = \sin(1),\quad y_0'(0) = 1,\ y_0'(1) = \cos(1) \tag{20b}$$

First Order Problem

$$y_1^{(iv)}(x, C_1) = (1 + C_1)\, y_0^{(iv)}(x) + (1 + C_1)\, g(x) - C_1 (y_0''(x))^2 \tag{21a}$$

$$y_1(0) = y_1(1) = y_1'(0) = y_1'(1) = 0 \tag{22b}$$

Second Order Problem:

$$y_2^{(iv)}(x, C_1, C_2) = (1 + C_1)\, y_1^{(iv)}(x, C_1) - 2C_1 y_0''(x)\, y_1''(x, C_1) + C_2 y_0^{(iv)}(x) + C_2 g(x) - C_2 (y_0(x))^2 \tag{23a}$$

$$y_2(0) = y_2(1) = y_2'(0) = y_2'(1) = 0 \tag{23b}$$

Solving problems (20a)-(23b) in succession we obtain the following the second order solution $\tilde{y} = y_0(x) + y_1(x, C_1) + y_2(x, C_1, C_2)$ C_1, C_2 can be easily calculated by the method of least squares as mention in the section 3.

Example 3: First, we apply OHAM to the following linear problem [18]

$$y^{(iv)}(x) = 1 - 4y(x), \quad -1 \le x \le 1 \tag{24a}$$

$$y(-1) = y(1) = 0,\quad y''(-1) = y''(1) = 0 \tag{24b}$$

Zeroth-Order Problem:

$$y_0^{(iv)}(x) + 4y_0(x) - 1 = 0 \tag{25a}$$

$$y_0(-1) = y_0(1) = 0,\quad y_0''(-1) = y_0''(1) = 0 \tag{25b}$$

First-Order Problem:

$$y_1^{(iv)}(x) + 4y_1(x) - y_0^{(iv)}(x) - 4y_0(x) + 1 = C_1 y_0^{(iv)}(x) + 4C_1 y_0(x) - C_1 \tag{26a}$$

$$y_1(-1) = y_1(1) = 0,\quad y_1''(-1) = y_1''(1) = 0 \tag{26b}$$

Second-Order Problem:

$$y_2^{(iv)}(x) + 4y_2(x) - y_1^{(iv)}(x) - 4y_1(x) = C_1 y_1^{(iv)}(x) + 4C_1 y_1(x) + C_2 y_0^{(iv)}(x) - C_2 + 4C_2 y_0(x) \tag{27a}$$

$$y_2(-1) = y_2(1) = 0,\quad y_2''(-1) = y_2''(1) = 0 \tag{27b}$$

and so on.
Thus the second order solution becomes

$$\tilde{y} = y_0(x) + y_1(x, C_1) + y_2(x, C_1, C_2)$$

C_1, C_2 can easily calculated be the method of least squares as mention in the section 3.

Example 4: Consider the following linear problem [18]

$$y^{(iv)}(x) = y(x) + y''(x) + x e^{x} - 4e^{x}, \quad 0 \le x \le 2 \tag{28a}$$

$$y_0(0) = 0, y_0'(0) = 1, \quad y_0(2) = 0, y_0'(2) = -e^{2} \tag{28b}$$

Like in Examle1, We Have Calculated As:

Example 5: Consider the non-linear problem [18]

$$y^{(iv)}(x) = y^{2}(x) + 1, \quad 0 \le x \le 2 \tag{29a}$$

$$y_0(0) = y_0'(0) = y_0(2) = y_0'(2) = 0 \tag{29b}$$

Zeroth-Order Problem:

$$y_0^{(iv)}(x) - 1 = 0, \tag{30a}$$

$$y_0(0) = y_0'(0) = y_0(2) = y_0'(2) = 0 \tag{30b}$$

First-Order Problem:

$$y_1^{(iv)}(x) - y_0^{(iv)}(x) + 1 = C_1 y_0^{(iv)}(x) - C_1 (y_0(x))^{2} - C_1 \tag{31a}$$

$$y_1(0) = y_1'(0) = y_1(2) = y_1'(2) = 0 \tag{32b}$$

Second-Order Problem:

$$y_2^{(iv)}(x, C_1, C_2) - y_1^{(iv)}(x, C_1) = C_1 y_1^{(iv)}(x, C_1) - 2C_1 y_0(x) y_1(x) + C_2 y_0^{(iv)}(x) - C_2 - C_2 (y_0(x))^{2} \tag{33a}$$

$$y_2(0) = y_2'(0) = y_2(2) = y_2'(2) = 0 \tag{33b}$$

and so on.

Thus the second order solution becomes

$$\tilde{y} = y_0(x) + y_1(x, C_1) + y_2(x, C_1, C_2)$$

C_1, C_2 can easily calculated be the method of least squares as mention in the section 3.

This problem has no exact solution but from the plot it is clear that the solution given by OHAM is completely agree with the analytic solution by VIM [18].

CONCLUSION

We applied a new powerful analytic method, OHAM for linear and nonlinear boundary value problems. We get highly accurate results by even first order approximation. This method provides us a convenient way to control the convergence and we can easily adjust the desired convergence regions. This approach is simple in applicability, as it does not require discretization or perturbation like other numerical and approximate methods. Moreover, this technique is fast converging to the exact solution and requires less computational work. This confirms our belief that the efficiency of the OHAM gives it much wider applicability. Mathematica software is used for symbolic derivations of some of the equations.

REFERENCES

1. Nayfeh, A.H. and D.T. Mook, 1979. Nonlinear Oscillations, John Wiley, New York.
2. Nayfeh, A.H., 1985. Problems in Perturbation, John Wiley, New York.
3. Nayfeh, A.H., 1981. Introduction to Perturbation Methods, John Wiley, New York.
4. Liao, S.J., 1992. Ph.D Dissertation, Jiao Tong University.
5. He, J.H., 2006. Homotopy Perturbation Method for solving boundary value problems, Phys. Lett. A, 350(1-2): 1187-1193.
6. He, J.H., 2005. Application of homotopy perturbation method to nonlinear wave equations, Chaos, Solitons Fractals, 26(3): 695.
7. He, J.H., 2004. Comparison of homotopy perturbation method and homotopy analysis method, Appl. Math. Comput., 156(2): 527.
8. He, J.H., 2004. Asymptotology by homotopy perturbation method, Appl. Math. Comput., 151(1): 591.
9. He, J.H., 2003. Homotopy perturbation method: a new nonlinear analytic technique, Appl. Math. Comput., 135(1): 73.
10. He, J.H., 1999. Homotopy Perturbation Technique, Comput. Methods Appl. Mech. Eng,. 178: 257-262.
11. Marinca, V., N. Herisanu and I. Nemes, 2008. An Optimal Homotopy Asymptotic Method with application to thin film flow, Cent. Eur. J. Phys., 6(3): 648-653.

12. Herisanu, N., V. Marinca, T. Dordea and G. Madescu, 2008. Proce. Romanian. Acad Series A, Maths. Phys. Tech. Sci. Infor. Sci., 9(3): 229-236.
13. Marinca, V., N. Herisanu, C. Bota and B. Marinca, 2009. Appl. Math. Lett., 22: 245-251.
14. Marinca, V. and N. Herisanu, 2008. Int. Comm. Heat and Mass Transfer, 35: 710-715.
15. Ali, J., S. Islam, Sirajul Islam and Gul Zaman, 2010. The solution of multipoint boundary value problems by the optimal homotopy asymptotic method. Computers and Mathematics with Applications, 59(6): 2000-2006.
16. Islam, S., Rehan Ali Shah and Ishtiaq Ali, 2010. Couette and Poiseuille flows for fourth grade fluid using Optimal homotopy asymptotic methods. Int. J. Non-Lin. Sci. Numeric. Simul., pp: 11-26.
17. Noor, M.A. and S.T. Mohyud-Din, 2007. An efficient method for fourth- order boundary value problems, Comput. and Math. with Applications, 54: 1101-1111.
18. Xu, L., 2009. The variational method for fourth order boundary value problems, Chaos, Solitons and Fractals, 39: 1386-1394.