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Design of Optimal Controllers for Linear Multivariable Dynamical Interval Systems

[1]S.M. Karbassi and [2]F. Soltanian

[1]Faculty of Mathematics, Yazd University, Yazd, 89195-741, Iran
[2]Department of Mathematics, Payame Noor University, Tehran, 19395-4697, Iran

Abstract: This paper studies robust performance analysis problem of linear multivariable uncertain systems. A new approach for the design of optimal controllers based on adjustment of parametric controllers to minimize the condition number of the eigenvector matrix of the closed-loop system is presented. First, a parametric formula for eigenvalue assignment of the desired spectrum for the system under nominal conditions is obtained and then these parameters are determined such that the condition number of the eigenvector matrix of the closed-loop matrix of the system is minimized. The time response of the uncertain system is then optimized. The results obtained are applied to an example and it is shown that the new results show better performance and the computations are less complicated and a more robust controller matrix for uncertain systems is obtained.

Key words: Interval systems · Uncertain systems · Optimal control · Eigenvalue assignment · Multivariable systems

INTRODUCTION

In actual situations, the structural parameters are often uncertain, such as the inaccuracy of the measurement, errors in the manufacturing process, incomplete knowledge of the physical system and so on. In many cases these parameter uncertainties lie within known bounds. Therefore, the uncertain systems play an important role in the control theory. The problem of robust pole assignment to these systems in such a way that the response of the system remains unchanged in the presence of disturbances, have attracted a great deal of attention in the past two decades. From mathematical point of view, robust pole assignment in uncertain systems is to obtain state feedback controllers such that for all admissible uncertainties, the change in the eigenvalue spectrum of the closed-loop system, in order to preserve the stability of the system under control, remains as small as possible. This problem has many applications in science and engineering, from the design of dynamical systems concerning buildings and bridges to the theory of microprocessors and robots [1]. The main purpose of this paper is to design robust controllers in the sense that not only the stability of the closed-loop system is preserved but also the time-response of the system under the presence of uncertain parameters is kept reasonable.

There are different types of uncertainties in the control systems. These uncertainties can appear in the elements of the matrix systems in state space representation, also due to ageing of some elements of the system which may appear unexpectedly. In this paper the former one will be considered by assuming that each uncertain parameter is allowed to vary within some known bounds and we will present a reliable and simple method to obtain the solution of robust pole assignment problem in uncertain systems. From 1970's onward the variable structure control remains one of the most interesting branches of control and the theory of variable structure control for various applicable systems with uncertainties has been developed by many investigators such as Emelyanov and Korovin [2], Utkin [3], Hsu and Lizarralde [4], Choi [5], Chen and Guo [6], Petersen [7] and see also [8-11]. One of the important methods proposed in this field, is due to Soylemez and Munro in 1997 [12-13] by introducing a concept called *pole coloring*. In their method a special performance index for every eigenvalue is minimized. The calculations of this method are rather complicated, also other present methods are quite difficult too and each method results in a different feedback matrix.

Several robust control system design methods for uncertain systems such as $H_\infty \rightarrow H_\infty$ approaches [14-17] and *LMI* approach [18-20] have been developed to solve this problem. But the method presented in this paper, is

Corresponding Author: S.M. Karbassi, Faculty of Mathematics, Yazd University, Yazd, 89195-741, Iran

very reliable and easy to apply. Here, first a general method of parametric eigenvalue assignment proposed by Karbassi and Tehrani [21] is used to provide a nonlinear system of equations for the nominal values system. It allows parameterizing explicitly the family of feedback gain matrices assigning a given spectrum. Then Genetic Algorithms (GAs) [22-26] and linear programming [27] are used in such a way as to find the best set of parameters of the parametric state feedback gain matrix such that the condition number of the closed-loop of the eigenvector matrix is minimized. Genetic algorithms (GAs) are directed random search techniques which can find the global optimal solution in complex multidimensional search spaces. In this problem GAs is appropriate because of the lack of prior knowledge about the properties of the spectral condition number as a function of the state feedback matrix. It is well known that, GAs is general search algorithms, based on the mechanics of natural evolution, which have been often used in system analysis in control problems. This choice is due to the ease of implementation and to the widely recognized ability of GAs to obtain robust controllers.

The rest of the paper is organized as follows. In section 2, the parameterization of state feedback controllers is reviewed. In section 3 the reliable and simple methodology is presented to obtain robust pole assignment state feedback controller. Section 4, demonstrates the efficiency and effectiveness of the procedure by an example. Conclusion is given in the last section.

Parameterization of State Feedback Controllers: In this section we briefly explain the method of Karbassi and Tehrani to determine the parametric state feedback matrix [21].

Consider the following system equations of a linear multivariable system in state space form:

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (1)$$
$$y(t) = Cx(t)$$

And State Feedback Control Law:

$$u(t) = Kx(t) \qquad (2)$$

Where $x(t)$ is the $n\times \to n \times 1$ state vector, $u(t)$ is the $m \times 1$ input vector, A and B are the $n \times n$ system matrix and the $n \times m$ input coefficient matrix, respectively and K is the $m \times n$ state feedback matrix. There exists a transformation matrix T such that transforms the system state vector to new space $x(t) = T\tilde{x}(t)$ and system matrices have the following form:

$$\tilde{A} = \begin{bmatrix} G_0 \\ I_{n-m}, 0_{n-m,m} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_0 \\ 0_{n-m,m} \end{bmatrix} \qquad (3)$$

Where G_0 is an $m \times n$ matrix and B_0 is an $m \times m$ upper triangular matrix. Now suppose that $\tilde{A}_\lambda$ is a parametric matrix such that its eigenvalues lie in the spectrum $\Lambda = \{\lambda_1,...,\lambda_n$ and has the vector companion form similar to $\tilde{A}$, i.e$\}$.

$$\tilde{A}_\lambda = \begin{bmatrix} G_\lambda \\ I_{n-m}, 0_{n-m,m} \end{bmatrix} \qquad (4)$$

thus as shown in [21]

$$K = B_0^{-1}(-G_0 + G_\lambda)T^{-1} \qquad (5)$$

is the feedback matrix which assigns the desired eigenvalue spectrum to the closed-loop system $A + BK$. If the characteristic polynomial of $\tilde{A}_\lambda$ has the form:

$$P_n(\lambda) = (-1)^n(\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + ... + c_n) \qquad (6)$$

then by equating the coefficients of $\det(\tilde{A}_\lambda - \lambda I) = 0$ with the characteristic polynomial (6), the following non-linear system of equations is obtained:

$$f_r(g_{ij}) = c_r, \qquad r = 1,2,...,n \qquad (7)$$

Where g_{ij} ($I = 1,..., m, j = 1,..., n$) are the elements of G_λ and by choosing $N = n(m - 1)$ unknowns arbitrarily, it is possible to solve the system (7). Thus we can obtain many number of state feedback matrices K, which assign the same eigenvalue spectrum Λ to the closed-loop system.

An important advantage of this method with respect to other parameterization methods, is that both linear and nonlinear relationships for state controller matrix can be generated.

Robust Pole Assignment Controller Design for Uncertain Systems: This study considers parametric model uncertainties with the following model description:

$$\dot{x}(t) = A(r)x(t) + B(r)u(t) \qquad (8)$$
$$y(t) = C(r)x(t)$$

Where r is the uncertainty vector lies within bounds $[r_{min}, r_{max}]$ and under nominal working conditions r_n.

As it is known, the spectral condition number of the eigenvector matrix of the closed-loop system matrix still remains as the most widely accepted measure of robustness. This is because of the following theorem, known as the Bauer-Fike theorem which gives an appropriate way to stabilize the uncertain systems:

Baure-Fike Theorem [28]: Let A be a square and diagonalizable matrix, that is there exists a nonsingular matrix X such that $X^{-1}AX$ is a diagonal matrix D. Then for an eigenvalue λ of the perturbed matrix $A + E$, we have where $\min|\lambda_i - \lambda| \leq \|X\|\|X^{-1}\|\|E\|$ is a matrix norm and λ_1, λ_2,...,λ_n are the eigenvalues of A.

For this reason, the spectral condition number $\kappa(X) = \|X\|\|X^{-1}\|$ provides a meaningful measure on the sensitivity of the closed-loop eigenvalues due to all suitably characterized perturbations in the parameter uncertainties.

Now, as there exists an infinite number of state feedback matrices for linear multivariable MIMO systems and correspondingly, each feedback matrix results in a different eigenvector matrix for the closed-loop system, therefore, a state feedback matrix K can be chosen such that the corresponding condition number of the eigenvector matrix of the closed-loop system is minimum. So, to solve the robust pole assignment problem for the uncertain system (8), first we use the method of parameterization reviewed in the previous section and find the parametric form of the state feedback matrix K for the following nominal system:

$$\begin{aligned} \dot{x}(t) &= A(r_n)x(t) + B(r_n)u(t) \\ y(t) &= C(r_n)x(t) \end{aligned} \tag{9}$$

To do this, first we find the nonlinear system of equations (7) for the nominal system (9), then by using genetic algorithm [22-26], we can select the linear or nonlinear relationship which produces the minimum condition number of the eigenvector matrix of the closed-loop system under nominal conditions respect to other relationships and therefore the convenient parametric state feedback matrix K can be determined.

The design process involves determining the set of parameters of the parametric state feedback gain matrix K which minimizes the spectral condition number $\kappa(X) = \|X\|\|X^{-1}\|$ as an objective function. For this purpose, we use linear programming [27] and determine the parameters of the state feedback gain matrix K such that the condition number of the eigenvector matrix of the uncertain closed-loop system is minimized subject to varying the uncertain parameters r_i in the specified intervals. Although this method is based on search techniques, however, employing the explicit parametric relationships enhances the maneuvering power.

The ability and effectiveness of the proposed method is shown by an example in the next section.

ILLASTRATIVE EXAMPLE

Let us consider the problem investigated by Soylemez and Munro [13] here, to be able to compare their results with our method. The problem is to find the state feedback controller matrix K for assigning the eigenvalue spectrum $\Lambda = \{-1 \pm 2i, -5\}$ to the uncertain system:

$$A = \begin{bmatrix} r_1 & r_2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}, B = \begin{bmatrix} r_3 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{10}$$

Where:

$$\begin{aligned} 0.8 &\leq r_1 \leq 1.2 \\ 1.7 &\leq r_2 \leq 2.3 \\ 0.95 &\leq r_3 \leq 1.05 \end{aligned} \tag{11}$$

Such that by variation of r_1, r_2, r_3 in the specified intervals, the difference between the eigenvalue spectrum of the closed-loop system of the uncertain system and the desired eigenvalue spectrum Λ, is minimized.

The nominal values of the uncertain parameters are:

$$\bar{r}_1 = 1 \quad , \quad \bar{r}_2 = 2 \quad , \quad \bar{r}_3 = 1 \tag{12}$$

By substituting these values into matrices A and B, the nominal value system is obtained. The characteristic polynomial for the closed-loop system under nominal working conditions with its roots lying in the spectrum Λ is:

$$P_3(\lambda) = \lambda^3 + 7\lambda^2 + 15\lambda + 25 \tag{13}$$

From [21], the nonlinear system of equations governing the nominal value system which assigns the eigenvalues given in spectrum Λ is:

$$\begin{cases} -(g_{11} + g_{22}) = 7 \\ g_{11}g_{22} - g_{12}g_{21} - g_{13} = 15 \\ g_{22}g_{13} - g_{12}g_{23} = 25 \end{cases} \tag{14}$$

By choosing the free parameter $g_{21} = \alpha$ and selecting $g_{11} = -4.6545$ and $g_{12} = 2.4055$ by genetic algorithm [22-26] such that the condition number of the closed-loop system under nominal conditions, is minimized, the value of the other parameters that satisfy the system of equations (14) can be chosen in terms of α. Substitution of these parameters in

$$G_\lambda = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \end{bmatrix} \tag{15}$$

Yields:

$$G_\lambda = \begin{bmatrix} -4.6545 & 2.4055 & -4.0829 - 2.4055\alpha \\ \alpha & -2.3455 & -6.4118 + 2.3455\alpha \end{bmatrix} \tag{16}$$

This results in the parametric state feedback matrix K_α:

$$K_\alpha = \begin{bmatrix} -13.3212 - 1.3333\alpha & -12.8951 - 1.7777\alpha & -25.7357 - 4.0663\alpha \\ \alpha & 1.3333\alpha - 2.6788 & 2.4485\alpha - 1.2399 \end{bmatrix} \tag{17}$$

Now by using the linear programming method [27], the parameter α is determined such that the condition number of the eigenvector matrix of the uncertain closed-loop system $A + BK_\alpha$ is minimized subject to varying the parameters r_i in the specified intervals. In this way, the optimal value $\alpha = -4.0188$ is obtained. By substituting this value into parametric matrix K_α, the appropriate controller matrix K is:

$$K = \begin{bmatrix} -7.9628 & -5.7509 & -9.3928 \\ -4.0188 & -8.0372 & -11.0802 \end{bmatrix} \tag{18}$$

Now to be able to compare this result with the controller matrix

$$K_s = \begin{bmatrix} -11.2476 & -33.0025 & -19.6451 \\ -1.6196 & -4.7523 & -2.8289 \end{bmatrix} \tag{19}$$

obtained by Soylemez and Munro [13], we define a new variable θ in the uncertain parameters r_i in the following form:

$$\begin{aligned} r_1 &= 0.8 + 0.4\theta \\ r_2 &= 1.4 + 0.6\theta \\ r_3 &= 0.95 + 0.1\theta \end{aligned} \tag{20}$$

In this way, varying θ from zero to one produces all r_i in their respective interval values. In Figure 1, the norm of the difference between the vector of perturbed eigenvalues and the vector of eigenvalues of the nominal

Plots of the variation of the norm of closed-loop system eigenvalues.

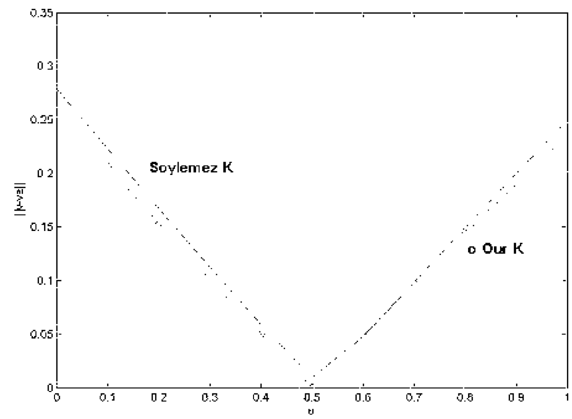

Fig. 1: r_1, r_2, r_3 Varying simultaneously

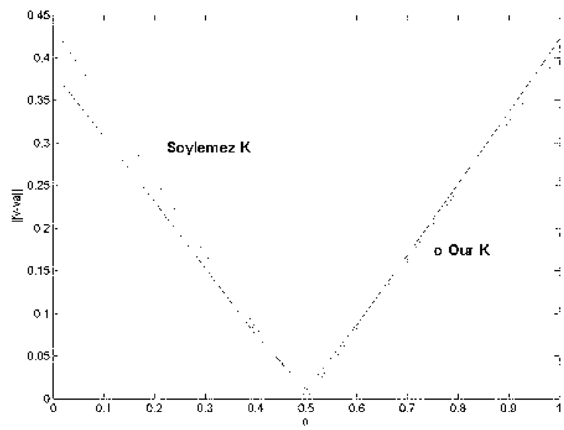

Fig. 2: r_1 Fixing and r_2, r_3 varying simultaneously

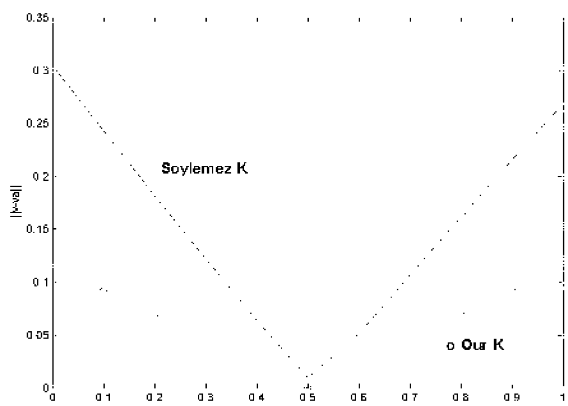

Fig. 3: r_2 Fixing and r_1, r_3 varying simultaneously

system when all the uncertain parameters vary, is shown for both cases. New values are shown with circles. Also, in Figures 2, 3 and 4, r_1, r_2 and r_3 are fixed while other parameters are varied, respectively. As it can be observed from these figures, the error obtained by the new method is less than the error obtained by the method

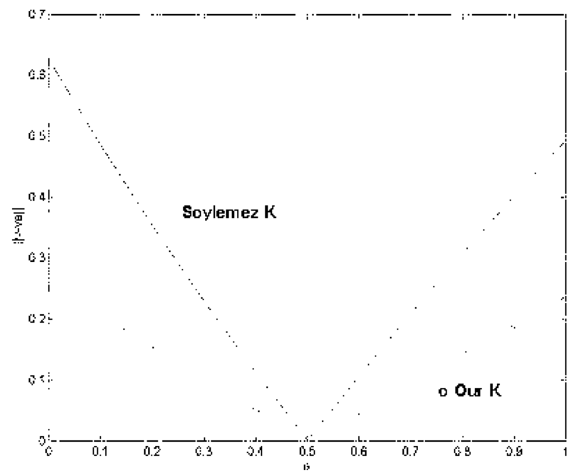

Fig. 4: r_3 Fixing and r_1, r_2 varying simultaneously

in [13]. It is interesting to note that the condition number of the eigenvector matrix of the closed-loop system is reduced from 29.6379 for the state feedback matrix K_s to 6.1623 for the matrix K obtained by our method. Also the Frobenius norm of K is 16.3284 while of K_s is 40.4328.

CONCLUSION

In this paper, a new method of obtaining robust state feedback matrix for uncertain systems is developed on the basis of a search technique (GAs) combined with linear programming.

Comparison of the results obtained with prior methods, shows that our method is easier and produces better results. Figures 1 and 2 compared with the result of Soylemez and Munro show small variations, but Figures 3 and 4 reveal quite a significant improvement in the results.

The obtained condition number value is in fact the minimum value that is found for the above system up to now. Thus the algorithm presented in this paper has produced a more robust feedback matrix in the sense that it has a lesser norm compared with previous result [13] and also a lesser condition number of the closed loop eigenvalues matrix. Further research to determine that this value is a local minimum or is a global minimum is under way. The work from here can be employed in other areas of uncertain systems concerning output feedback, time-delay systems, tracking systems as well as discrete-time systems.

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