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Pro-C Completions Process of Crossed Squares and Some Relations

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Abstract: In this paper we will examine the Pro-C completion process of a crossed square. For a crossed square (L, M, N, P), we will determine the relation between the completion (L, M, N, P) and the completion of the individual pieces of the given structure.

Key words: Pro-C crossed module . pro-C completion . pro-C crossed square . cat^1-group

INTRODUCTION

Pro-C groups occur in problems relating to Number Theory, Algebraic Geometry and Algebraic Topology. Although the category of pro-C groups forms a natural extension of the category of finite groups, it carries a richer structure in that it has categorical objects and constructions which do not exist in finite case; e.g. projective limits and free products. The existence of such constructions in the extended category leads to the definition of profinite analogues of the usual constructions of combinatorial group theory such as free groups and presentations of a group by generators and relations.

The theory of crossed modules becomes a useful tool in combinatorial and cohomological group theory. Pro-C crossed modules had been firstly defined by F.J. Korkes and T. Porter in [1]. They also examine the relationship between crossed modules and pro-C crossed modules for C a full class of finite groups (c.f. [1]).

In this paper up to the work [1] we will introduce the pro-C analogues of crossed square and cat²-group. Also we will give the pro-C completion process of the crossed squares by using a similar method given in [1].

TERMINOLOGY

In this paper C will denote a class of finite groups which is closed under the formation of subgroups, homomorphic images, finite products and which contains at least one non-trivial group. Pro-C groups are profinite groups whose finite quotients are in C.

The class C will be assumed to be full in the sense that C must also be closed under extension of groups. Of particular interest are the examples where C is the class of finite p-groups or finite solvable groups.

CROSSED MODULES, CROSSED SQUARES, CAT²-GROUPS AND THEIR PRO-C ANALOGUES

Throughout this paper we denote an action of P on M by $p\cdot m = {}^{p}m$.

Crossed modules

Definition 1.1: A crossed module is a group homomorphism ∂:M$\to$P together with an action of P on M satisfying

$$(C1)\ \partial({}^{p}m) = p\partial(m)p^{-1}$$

and

$$(C2)\ {}^{\partial m}m' = mm'm^{-1}$$

for all $m, m' \in M$, $p \in P$.

This second condition is called the Peiffer identity. We will denote such a crossed module by (M,P,∂).

Example 1.2: For H a normal subgroup of P, the inclusion homomorphism i:H$\to$P makes (H,P,i) into a crossed module where P acts on H by conjugation.

Example 1.3: If M is a P-module and 0:M$\to$P is the zero homomorphism, then (M,P,0) is a crossed module.

Definition 1.4: A morphism of crossed modules from (M, P, ∂) to (M′, P′, ∂') is a pair (ϕ, φ) of group homomorphisms,

$$\phi : M \to M',\ \varphi : P \to P'$$

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such that

$$\phi({}^{p}m) = {}^{\varphi(p)}\phi(m) \text{ and } \partial'\phi(m) = \varphi\partial(m).$$

hus we get a category XMod of crossed modules. There are special classes of morphisms, those in which $P = P'$ and ψ is the identity morphism. For fixed P, such a morphism

$$(\phi, id_P) : (M, P, \partial) \to (M', P, \partial')$$

will be called a morphism of crossed modules over P. These give a subcategory Xmod/P of XMod.

Definition 1.5: Let (M, P, ∂) be a crossed module. (M_1, P_1, ∂_1) is a subcrossed module of (M, P, ∂) if

- M_1 is a subgroup of M and P is a subgroup of P,
- ∂_1 is the restriction of ∂ to M_1 and
- the action of P_1 on M_1 is induced by the action of P on M.

A subcrossed module (M_1, P_1, ∂_1) of (M, P, ∂) is a normal subcrossed module if

- P_1 is a normal subgroup of P,
- ${}^{p}m_1 \in M_1$ for all $p \in P$, $m_1 \in M_1$ and
- ${}^{p_1}m \cdot m^{-1} \in M_1$ for all $p_1 \in P_1$, $m \in M$

Definition 1.6 (Pro-C crossed modules) A pro-C crossed module (M, P, ∂) is a crossed module in which M and P are pro-C groups, ∂ is a continuous homomorphism and the left P-action on M is a continuous P-action.

Example 1.7: For H a closed normal subgroup of P, the continuous inclusion homomorphism $i: H \to P$ makes (H, P, i) into a pro-C crossed module where P acts on H by conjugation continuously.

Definition 1.8: A morphism

$$(\phi, \varphi) : (M, P, \partial) \to (M', P', \partial')$$

of pro-C crossed modules is a morphism of the underlying crossed modules in which both ϕ and φ are continuous homomorphisms of pro-C groups. This gives us categories "Pro-C. XMod" and "Pro-C. Xmod/P" for P a pro-C group and also a functor

U_{XMod}:Pro-C.XMod$\to$XMod

which forgets the topological structure.

Recall the corresponding situation for groups; the forgetful functor

$$U_{Grps}: \text{Pro-C.Grps} \to \text{Grps}$$

(in the hopefully obvious notation) has a left adjoint, known as the pro-C completion functor, which we will denote by a.

This is defined as follows:

If P is a group, let Ω (P) be the directed set of normal finite index subgroup W of P with $P/W \in C$, then

$$P = \varprojlim_{W \in \Omega(P)} P/W$$

We will sometimes write $W_{fin} \lhd P$ as indicating that $W \in \Omega(P)$.

Crossed squares: The following definition is due to D. Guin-Walery and J.-L. Loday [2].

Definition 1.9: A crossed square of groups is a commutative square of groups

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"] & P \end{tikzcd}$$

together with an action of P on L, M and N. There are thus actions of N on L and M via μ' and M acts on L and N via μ and a function $h: M \times N \to L$ such that, for all $l \in L$, $m_1, m \in M$, $n_1, n \in N$ and $p \in P$ the following axioms hold: the homomorphisms λ, λ', μ, μ' and $k = \mu\lambda = \mu'\lambda'$ are crossed modules for the corresponding actions and the morphisms of maps $(\lambda) \to (k)$; $(k) \to (\mu)$; $(\lambda') \to (k)$ and $(k) \to (\mu')$ are morphisms of crossed modules,

$$\lambda h(m,n) = m^{\mu'(n)}m^{-1} \text{ and } \lambda' h(m,n) = {}^{\mu'(m)}nn^{-1}$$

$$h(\lambda(l),n) = l^{n}l^{-1} \text{ and } h(m,\lambda'(l)) = {}^{m}ll^{-1}$$

$$h(m_1m_2,n) = {}^{m_1}h(m_2,n)h(m_1,n)$$

and

$$h(m,n_1n_2) = h(m,n_1)^{n_1}h(m,n_2)$$

$$h({}^{p}m, {}^{p}n) = {}^{p}h(m,n)$$

$${}^{m}({}^{n}l)h(m,n) = h(m,n)^{n}({}^{m}l)$$

for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $l \in L$.

Example 1.10: Let P be a group. Let M, N be normal subgroups of P, μ and μ' are normal subgroup inclusions and $L = M \cap N$, with h being the conjugation map.

$$\begin{tikzcd} M \cap N \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"] & P \end{tikzcd}$$

Definition 1.11: A morphism of crossed square is a commutative diagram

$$\begin{tikzcd} & L_1 \arrow[rr, "\lambda_1"] \arrow[d, "\lambda_1'"] \arrow[ddl, "\Phi_1"'] & & M_1 \arrow[d, "\mu_1"] \arrow[ddl, "\Phi_2"] \\ & N_1 \arrow[rr, "\mu_1'"] \arrow[ddl, "\Phi_3"] & & P_1 \arrow[ddl, "\Phi_4"] \\ L_2 \arrow[rr, "\lambda_2"] \arrow[d, "\lambda_2'"'] & & M_2 \arrow[d, "\mu_2"'] & \\ N_2 \arrow[rr, "\mu_2'"] & & P_2 & \end{tikzcd}$$

$$\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4) : (L_1, M_1, N_1, P_1) \to (L_2, M_2, N_2, P_2)$$

consisting of homomorphisms

$$\Phi_1 : L_1 \to L_2 \qquad \Phi_1 : M_1 \to M_2$$
$$\Phi_3 : N_1 \to N_2 \qquad \Phi_4 : P_1 \to P_2$$

such that the cube of homomorphisms commutative

$$\Phi_1 h(m_1, n_1) = h(\Phi_2(m_1), \Phi_3(n_1))$$

With $m_1 \in M_1$, $n_1 \in N_1$ and each of the homomorphisms Φ_1, Φ_2, Φ_3 is Φ_4-equivariant. The category of crossed squares will be denoted by Crs^2. There are special classes of morphisms, those in which $P_1 = P_2$ and Φ_4 is the identity morphism. For fixed P, such a morphism $\Phi = (\Phi_1, \Phi_2, \Phi_3, id): (L_1, M_1, N_1, P_1) \to (L_2, M_2, N_2, P_1)$ will be called a morphism of crossed squares over P_1. These give a subcategory Crs^2/P of Crs^2.

Definition 1.12: A crossed square (L_1, M_1, N_1, P_1)

$$\begin{tikzcd} L_1 \arrow[r, "\lambda_1"] \arrow[d, "\lambda_1'"'] & M_1 \arrow[d, "\mu_1"] \\ N_1 \arrow[r, "\mu_1'"] & P_1 \end{tikzcd}$$

is a subcrossed square of (L, M, N, P)

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"] & P \end{tikzcd}$$

if

- L_1 is a subgroup of L, M_1 is a subgroup of M, N_1 is a subgroup of N, P_1 is a subgroup of P,
- λ_1 is restriction of λ to L_1, μ_1 is restriction of μ to M_1, λ_1' is restriction of λ' to L_1, μ_1' is restriction of μ' to N_1,
- actions of P_1 on L_1, M_1 and N_1 are induced by the actions of P on L, M and N, respectively.
- $h_1: M_1 \times N_1 \to L_1$ is the restriction of $h: M \times N \to L$ to $M_1 \times N_1$

Definition 1.13: A subcrossed square (L_1, M_1, N_1, P_1)

$$\begin{tikzcd} L_1 \arrow[r, "\lambda_1"] \arrow[d, "\lambda_1'"'] & M_1 \arrow[d, "\mu_1"] \\ N_1 \arrow[r, "\mu_1'"] & P_1 \end{tikzcd}$$

is a normal subcrossed square of (L, M, N, P)

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"] & P \end{tikzcd}$$

if

- P_1 is a normal subgroup of p, M_1 is a normal subgroup of M, N_1 is a normal subgroup of N,
- ${}^{p}l_1 \in L_1$ for all $p \in P$, $l_1 \in L_1$
  ${}^{p}m_1 \in M_1$ for all $p \in P$, $m_1 \in M_1$
  ${}^{p}n_1 \in N_1$ for all $p \in P$, $n_1 \in N_1$
- ${}^{p}ll_1 \in L_1$ for all $p_1 \in P_1$, $l \in L$
  ${}^{p1}mm^{-1} \in M_1$ for all $p_1 \in P_1$, $m \in M$
  ${}^{p1}nn^{-1} \in N_1$ for all $p_1 \in P_1$, $n \in N$

Let (L,M,N,P)

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"] & P \end{tikzcd}$$

be a crossed square and (L_1, M_1, N_1, P_1)

$$\begin{tikzcd} L_1 \arrow[r, "\lambda_1"] \arrow[d, "\lambda_1'"'] & M_1 \arrow[d, "\mu_1"] \\ N_1 \arrow[r, "\mu_1'"] & P_1 \end{tikzcd}$$

be a normal subcrossed square of (L, M, N, P). Let $\overline{\lambda}$ is induced by λ, $\overline{\mu}$ is induced by μ, $\overline{\lambda'}$ is induced by λ', $\overline{\mu'}$ is induced by μ'. Then there is an action of P/P_1 on L/L_1, M/M_1 and N/N_1 given by:

$$
\begin{aligned}
p^{P_1}(lL_1) &= ({}^{p}l)L_1 \\
p^{P_1}(mM_1) &= ({}^{p}m)M_1 \\
p^{P_1}(nN_1) &= ({}^{p}n)N_1
\end{aligned}
$$

and thus there are actions of N/N_1 on L/L_1 and M/M_1 via $\overline{\mu'}$

$$
\begin{aligned}
{}^{nN_1}(lL_1) &= {}^{\overline{\mu'}(nN_1)}(lL_1) = {}^{\mu'(n)}(lL_1) = ({}^{\mu'(n)}l)L_1 \\
{}^{nN_1}(mM_1) &= {}^{\mu'(nN_1)}(mM_1) = {}^{\mu'(n)P_1}(mM_1) = ({}^{\mu'(n)}m)M_1
\end{aligned}
$$

and M/M_1 acts on L/L_1 and N/N_1 via $\overline{\mu}$

$$
\begin{aligned}
{}^{mM_1}(lL_1) &= {}^{\overline{\mu}(mM_1)}(lL_1) = {}^{\mu(m)P_1}(lL_1) = ({}^{\mu(m)}l)L_1, \\
{}^{mM_1}(nN_1) &= {}^{\overline{\mu}(mM_1)}(nN_1) = {}^{\mu(m)P_1}(nN_1) = ({}^{\mu(m)}n)N_1
\end{aligned}
$$

The conditions for (L_1, M_1, N_1, P_1) to be normal in (L, M, N, P) ensure that the actions are well defined. Let h_1: $M_1\times N_1\to L_1$ is defined by $h(mM_1, nN_1) = h(m, n)L_1$. It is clear that ($L/L_1$, M/M_1, N/N_1, P/P_1,)

$$
\begin{tikzcd}
L/L_1 \arrow[r, "\overline{\lambda}"] \arrow[d, "\overline{\lambda'}"'] & M/M_1 \arrow[d, "\overline{\mu}"] \\
N/N_1 \arrow[r, "\overline{\mu'}"] & P/P_1
\end{tikzcd}
$$

is a crossed square. It is called the quotient crossed square of (L, M, N, P) by (L_1, M_1, N_1, P_1) and denoted by (L, M, N, P)/(L_1, M_1, N_1, P_1).

Definition 14 (Pro-C crossed squares) A pro-C crossed square (L, M, N, P)

$$
\begin{tikzcd}
L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\
N \arrow[r, "\mu'"] & P
\end{tikzcd}
$$

is a crossed square in which L, M, N and P are pro-C groups, λ, λ', μ, μ' are continuous homomorphisms, all the actions are continuous and the h-map is continuous. A morphism

$$
\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4): (L_1, M_1, N_1, P_1) \to (L_2, M_2, N_2, P_2)
$$

of pro-C crossed squares is a morphism of the underlying crossed squares such that the maps Φ_1, Φ_2, Φ_3 and Φ_4 are continuous morphisms of pro-C groups. This gives us categories "Pro-C.Crs² and " Pro-C.Crs² /P" for P a pro-C group and also a functor

$$
U_{Crs}{}^2: \text{Pro-C.Crs}^2 \to \text{Crs}^2
$$

which forgets the topology.
We wish to see if the forgetful functor

$$
U_{Crs}{}^2: \text{Pro-C.Crs}^2 \to \text{Crs}^2
$$

also has a left adjoint. The obvious approach using some idea of "normal" subcrossed complex of finite index is technically messy so we use an equivalence formulation involving Loday's notion of cat²-groups as used in [1].

CAT¹-GROUPS, CAT²-GROUPS, THEIR PRO-C ANALOGUE AND THE COMPLETION PROCESS

Cat¹-groups

Definition 2.1 A cat¹-group is a triple (G, s, t), where G is a group and s,t are endomorphisms of G satisfying conditions

$$
\begin{gathered}
st = t \text{ and } ts = s \\
[\mathrm{Ker}s, \mathrm{Ker}t] = 1
\end{gathered}
$$

There is an obvious notion of a morphism between cat¹-groups: if (G, s, t) and (G′, s′, t′) are cat¹-groups a morphism

$$
\phi: (G,s,t) \to (G,s',t')
$$

is a group homomorphism ϕ:G→G such that

$$
\begin{aligned}
s'\phi &= \phi s \\
t'\phi &= \phi t
\end{aligned}
$$

This gives a category, which we will denote Cat¹(Grps), of cat¹-groups and morphisms between them.

In [3] Loday shows that there is an equivalence between the categories XMod and Cat¹(Grps). This equivalence is constructed as follows:

Given ∂:C→R, a crossed module, we form the semi-direct product, G = C⋊R, using the action of R on C. The structural maps s, t are given by

$$
s(c, r) = (1, r) \quad \text{and} \quad t(c, r) = (1, \partial(c)r)
$$

for $c \in C$, $r \in R$. This clearly satisfies the axioms for a cat¹-group. On the otherhand given a cat¹-group (G, s, t), we set C = Kers, R = Im s and $\partial = t|_C$, the restriction of t to C. The action of R on C is by conjugation within G. again axioms are easily checked.

Cat²-Groups

Definition 2.2: A cat²-group is a 5-tuble (G, s_1, t_1, s_2, t_2) where (G, s_i, t_1), i = 1,2, are cat¹-groups and

$$s_is_j = s_js_i,\ \ t_it_j = t_jt_i,\ \ s_it_j = t_js_i$$

for i, j = 1,2, $i \neq j$.

There is an obvious notion of a morphism between cat²-groups: if (G, s_1, t_1, s_2, t_2) and $(G, s_1', t_1', s_2', t_2')$ are cat²-groups a morphism

$$\phi : (G, s_1, t_1, s_2, t_2) \to (G, s_1', t_1', s_2', t_2')$$

is a group homomorphism $\phi: G \to G$ such that

$$s_1'\phi = \phi s_1$$
$$t_1'\phi = \phi t_1$$

and

$$s_2'\phi = \phi s_2$$
$$t_2'\phi = \phi t_2$$

This gives a category, which we will denote Cat²(Grbs), of cat²-groups and morphisms between them.

Cat¹-pro-C groups: We next introduce the pro-C analogue of the above.

Definition 2.3: A cat¹-pro-C group is a cat¹-group (G, s, t) in which G is a pro-C group and s and t are continuous endomorphisms of G.

A morphism of cat¹-pro-C group is a morphism

$$\phi : (G,s,t) \to (G,s',t')$$

of the underlying cat¹-groups such that ϕ is a continuous morphism of pro-C groups. This gives a category of cat¹- pro-C groups that we will denote Cat¹(Pro-C.Grps). There is a forgetful functor from Cat¹(Pro-C.Grps) to Cat¹(Grps) which will be denoted by U_{CGrps}.

Cat²-pro-C groups

Definition 2.4: A cat²-pro-C group is a cat²-group (G, s_1, t_1, s_2, t_2) in which G is a pro-C group and s_1, s_2, t_1 and t_2 are continuous endomorphisms of G.

A morphism of cat²-pro-C group is a morphism

$$\phi : (G, s_1, t_1, s_2, t_2) \to (G, s_1', t_1', s_2', t_2')$$

of the underlying cat²-groups such that ϕ is a continuous morphism of pro-C groups. This gives a category of cat²-pro-C groups that we will denote Cat²(Pro-C.Grps). There is a forgetful functor from Cat²(Pro-C.Grps) to Cat²(Grps) which will be denoted by $U_C{}^2{}_{Grps}$.

PRO-C COMPLETION OF CROSSED SQUARES

Lemma 3.1: (procompletion) There is an equivalence of categories

$$\text{Pro-C. XMod} \xrightarrow{\cong} \text{Cat}^1(\text{Pro-C.Grps})$$

compatible, via the forgetful functors, with the equivalence between XMod and Cat¹(Grps) i.e. the diagram

$$\begin{tikzcd}
\text{Pro-}\mathcal{C}.\mathfrak{XMod} \arrow[r, "\cong"] \arrow[d, "\mathcal{U}_{\mathfrak{XMod}}"] & \mathfrak{Cat}^1(\text{Pro-}\mathcal{C}.\mathbf{Grps}) \arrow[d, "\mathcal{U}_{\mathrm{CGrps}}"] \\
\mathfrak{XMod} \arrow[r, "\cong"] & \mathfrak{Cat}^1(\mathbf{Grps})
\end{tikzcd}$$

commutes.

Proof: In fact, if (C, R, ∂) is a pro-C crossed module, then G = RoC is a pro-C group and the endomorphisms s and t, given earlier, are continuous, so resulting (G, s, t) is a cat¹-pro-C group. Similarly if (G, s, t) is a cat¹-pro-C group then $(\text{Kers}, \text{Im } s, t|_{\text{Kers}})$ is a pro-C crossed module.

Theorem 3.2: ([3]) There is equivalence of categories between the category of cat²-groups and that of crossed squares.

Proof: Let (G, s_1, t_1, s_2, t_2) be a cat²-group. Define $L = \text{Kers}_1 \cap \text{Kers}_2$, $M = \text{Ims}_1 \cap \text{Kers}_2$, $N = \text{Kers}_1 \cap \text{Ims}_2$, $P = \text{Kers}_1 \cap \text{Kers}_2$ and λ=restriction of t_1 to L, λ'=restriction of t_2 to L, μ'=restriction of t_1 to M, μ=restriction of t_2 to N. If m is in M and $n \in N$ is in N then the commutator [m,n] is in L therefore the function $h: M \times N \to L$, h(m,n) = [m,n] is well defined. The equality $\mu\lambda = \mu'\lambda'$ follows from $t_1t_2 = t_2t_1$. Using the equivalence of cat¹-groups with crossed modules we easily prove axiom (i) of the definition of crossed square. The other axioms are also easily verified: It suffices to compute in G, replacing h(m,n) by commutator and all the actions by conjugation.

$$\begin{tikzcd}
\operatorname{Ker} s_1 \cap \operatorname{Ker} s_2 \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & Im s_1 \cap \operatorname{Ker} s_2 \arrow[d, "\mu"] \\
\operatorname{Ker} s_1 \cap Im s_2 \arrow[r, "\mu'"] & Im s_1 \cap Im s_2
\end{tikzcd}$$

We will now construct a cat²-group from a

$$\begin{tikzcd}
L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\
N \arrow[r, "\mu'"] & P
\end{tikzcd}$$

crossed square. First there are semi-direct product LoN and MoP. Define an action of LoM on LoN as follows:

$$^{(m,p)}(l,n) = {}^{m}({}^{p}l, {}^{p}n) = ({}^{\mu(m)p}l\,h(m, {}^{p}n), {}^{p}n)$$

Use of the axiom (iv), (v) and (vi) of def. of. crossed square shows that this action is well defined. Put

$$G = (LoN)o(MoP)$$

s_1=projection on (MoP) and define t_1 by

$$t_1(l,n,m,p) = ({}^{\lambda(l)\mu(n)}m, {}^{\mu'(n)}p)$$

Then (G, s_1, t_1) is a cat¹-group.

We can switch the role of M and N, that is we can define an action of NoP on LoMsuch that G is canonically isomorphic to (LoM)o(NoP). These two categorical group structures on G commute because $\mu\lambda = \mu'\lambda'$. Thus we have constructed a cat²-group.

Theorem 3.3 ([3]) There is an equivalence of categories

$$\text{Pro-C. Crs}^2 \xrightarrow{\cong} \text{Cat}^2(\text{Pro-C.Grps})$$

compatible, via the forgetful functors, with the equivalence between Crs² and Cat²(Grps) i.e. the diagram

$$\begin{tikzcd}
\mathbf{Pro}-\mathcal{C}.\mathfrak{Crs}^2 \arrow[r, "\cong"] \arrow[d, "U_{\mathfrak{Crs}^2}"] & \mathfrak{Cat}^2(\mathbf{Pro}\text{-}\mathcal{C}.\mathbf{Grps}) \arrow[d, "U_{C^2\mathbf{Grps}}"] \\
\mathfrak{Crs}^2 \arrow[r, "\cong"] & \mathfrak{Cat}^2(\mathbf{Grps})
\end{tikzcd}$$

commutes.

Proof: In fact, if

$$\begin{tikzcd}
L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\
N \arrow[r, "\mu'"] & P
\end{tikzcd}$$

is a pro-C crossed square, then G = (LoM)o(NoP) is a pro-C group and the endomorphisms s_1, s_2, t_1 and t_2, given earlier, are continuous, so resulting (G, s_1, t_1, s_2, t_2) is a cat²-pro-C group. Similarly if (G, s_1, t_1, s_2, t_2) is a cat²-pro-C group then

$$\begin{tikzcd}
\operatorname{Ker} s_1 \cap \operatorname{Ker} s_2 \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & Im s_1 \cap \operatorname{Ker} s_2 \arrow[d, "\mu"] \\
\operatorname{Ker} s_1 \cap Im s_2 \arrow[r, "\mu'"] & Im s_1 \cap Im s_2
\end{tikzcd}$$

is a pro-C crossed square.

This lemma will enable us to prove the existence of a left adjoint for

$$U_{Crs}{}^2\colon \text{Pro-C.Crs}^2 \to \text{Crs}^2$$

by constructing one for

$$U_C{}^2{}_{Grps}\colon \text{Cat}^2(\text{Pro-C.Grps}) \to \text{Cat}^2(\text{Grps})$$

This latter construction will need projective limit within Cat²(Pro-C.Grps) and so we will briefly look at their construction as it sheds more light on the pro-C completion functor that will result from their use.

As a consequence of [1]; given a projective system F:I→Cat²(Grps), one notes that F is a projective system of groups together with four endomorphisms of projective systems, s_1, s_2, t_1, t_2: F→F satisfying

$$s_i s_j = s_j s_i,\ t_i t_j = t_j t_i,\ s_i t_j = t_j s_i$$

for i, j = 1,2, i≠j and $[\operatorname{Ker} s_1, \operatorname{Ker} t_1] = 1$, $[\operatorname{Ker} s_2, \operatorname{Ker} t_2] = 1$. We form $\underline{\lim}$ F by taking the limit of this underlying system of pro-C groups together with the induced endomorphism $\underline{\lim}$ s and $\underline{\lim}$ t. Writing the result as $(\overline{F}, \overline{s}_1, \overline{t}_1, \overline{s}_2, \overline{t}_2)$, we have merely to check the commutator conditions $[\operatorname{Ker}\overline{s}_1, \operatorname{Ker}\overline{t}_1] = 1$ and $[\operatorname{Ker}\overline{s}_2, \operatorname{Ker}\overline{t}_2] = 1$. However $\overline{F}$ can be realized as a subgroup of the product $\prod_{i\in I} F(i)$ and $\overline{s}_1((x_i)) = (s_1(i)x_i)$, $\overline{t}_1((x_i)) = (t_1(i)x_i)$ similarly for $\overline{s}$ and $\overline{t}$, so as the commutator subgroups $[\operatorname{Ker} s_1, \operatorname{Ker} t_1]$ and $[\operatorname{Ker} s_2, \operatorname{Ker} t_2]$ are trivial for each i in I, it is so for the limit as it can be calculated "pointwise".

Proposition 3.4: A pro-C completion functor from Cat²(Grps) to Cat²(Pro-C.Grps) exists, (i.e. the forgetful functor $U_{C^2 \text{Grps}}$ has a left adjoint).

Proof: An exact sequence

$$1 \to (K, s_1', t_1', s_2', t_2') \xrightarrow{u} (G, s_1, t_1, s_2, t_2) \xrightarrow{v} (H, s_1'', t_1'', s_2'', t_2'') \to 1$$

of cat²-groups is an exact sequence

$$1 \to K \to G \to H \to I$$

of the underlying groups and continuous maps compatible with the source and target maps. In this situation, we say that the cat²-group (H, s_1, ″, t_1'', s_2'', t_2'') is the quotient of (G, s_1, t_1, s_2, t_2) by the normal sub-cat²-group (K, s_1', t_1', s_2', t_2'). The later is of finite index in (G, s_1, t_1, s_2, t_2) if H is finite.

Given any cat²-group (G, s_1, t_1, s_2, t_2) the set of its normal sub-cat²-group (N, s_1', t_1', s_2', t_2') of finite index with $G/N \in C$ is directed by the inclusion so we can form an inverse system of finite quotient of (G, s_1, t_1, s_2, t_2) and can take its limit within the category of cat²-pro-C groups. (As usual one considers each finite cat²-group as a pro-C one having the discrete topology.)
Thus we define a pro-C completion functor:

$$\text{Cat}^2(\text{Grps}) \to \text{Cat}^2(\text{Pro-C.Grps})$$

by (G, s_1, t_1', s_2, t_2) = $\underline{\lim}$ {finite quotients of (G, s_1, t_1, s_2, t_2) by (N, s_1', t_1', s_2', t_2') }. General consideration of category theory then imply that this functor is left adjoint to the forgetful functor from Cat²(Pro-C.Grps) to Cat²(Grps).

Proposition 3.5: A pro-C completion functor from Crs² to Pro-C.Crs² exists, (i.e. the forgetful functor U_{Crs^2} has a left adjoint).

Proof: In the diagram

$$\begin{tikzcd} \text{Pro}-\mathcal{C}.\mathfrak{Crs}^2 \arrow[r, "\cong"] \arrow[d, "\mathcal{U}_{\mathfrak{Crs}^2}"] & \mathfrak{Cat}^2(\text{Pro-}\mathcal{C}.\text{Grps}) \arrow[d, "\mathcal{U}_{C^2\text{Grps}}"] \\ \mathfrak{Crs}^2 \arrow[r, "\cong"] & \mathfrak{Cat}^2(\text{Grps}) \end{tikzcd}$$

we have found a left adjoint to the (vertical) functor on the right. This induces via the equivalence of categories, a left adjoint for the left hand (vertical) functor.

Notation 3.6: We will denote by (L, M, N, P) or less accurately, (L, M, N, P),

$$\begin{tikzcd} \widetilde{L} \arrow[r, "\widetilde{\lambda}"] \arrow[d, "\widetilde{\lambda'}"'] & \widetilde{M} \arrow[d, "\widetilde{\mu}"] \\ \widetilde{N} \arrow[r, "\widetilde{\mu'}"] & \widetilde{P} \end{tikzcd}$$

the pro-C completion of the crossed square (L, M, N, P)

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"] & P \end{tikzcd}$$

It is natural to want to compare this pro-C completion (L, M, N, P) with the pro-C completions $\hat{L}$,M,N,P and $\hat{\lambda}$, μ, λ', μ' of the individual pieces of data involved. One may even wonder why $\hat{L}$,M,N,P

$$\begin{tikzcd} \widehat{L} \arrow[r, "\widehat{\lambda}"] \arrow[d, "\widehat{\lambda'}"'] & \widehat{M} \arrow[d, "\widehat{\mu}"] \\ \widehat{N} \arrow[r, "\widehat{\mu'}"] & \widehat{P} \end{tikzcd}$$

is not itself always the same as (L, M, N, P),

$$\begin{tikzcd} \widetilde{L} \arrow[r, "\widetilde{\lambda}"] \arrow[d, "\widetilde{\lambda'}"'] & \widetilde{M} \arrow[d, "\widetilde{\mu}"] \\ \widetilde{N} \arrow[r, "\widetilde{\mu'}"] & \widetilde{P} \end{tikzcd}$$

To start the study of this problem we first look at P.

Proposition 3.7: For any crossed square Proposition 3.7, P≅P.

Proof: This follows from an adjoint functor argument: There is a forgetful functor $R:\text{Crs}^2 \to \text{Grps}$
given by R (L, M, N, P) = P also an analogous one

$$R_{pC}: \text{Pro-C.Crs}^2 \to \text{Pro-C.Grps}$$

These have left adjoints L and L_{pC} defined by L (P) = (P, P, P, P)

$$\begin{tikzcd} P \arrow[r, "id"] \arrow[d, "id"'] & P \arrow[d, "id"] \\ P \arrow[r, "id"] & P \end{tikzcd}$$

with the h-map given by h (m, n) = [m, n] and similarly for L_{pC}.

We have a diagram of left and right adjoints

$$\begin{tikzcd}
\mathbf{Pro}-\mathcal{C}.\mathfrak{Crs}^2 \arrow[r, shift left, "R_p"] \arrow[d, shift left, "\mathcal{U}_{\mathfrak{Crs}^2}"] & \mathfrak{Cat}^2(\mathbf{Pro}-\mathcal{C}.\mathbf{Grps}) \arrow[l, shift left, "L_p"] \arrow[d, shift left, "\mathcal{U}^2_{\mathrm{C}}\mathbf{Grps}"] \\
\mathfrak{Crs}^2 \arrow[u, shift left, "(\sim)"] \arrow[r, shift left, "R"] & \mathfrak{Cat}^2(\mathbf{Grps}) \arrow[l, shift left, "L"] \arrow[u, shift left, "(\wedge)"]
\end{tikzcd}$$

The right adjoint diagram commutes, so the left adjoint diagram commutes up to isomorphism, i.e. (P, P, P, P) (P, P, P, P) but better we have a sequence of isomorphisms: for a pro-C group H,

$$\begin{aligned}
&\mathrm{Pro}-\mathrm{C.Grps}(R_{pC}(L,M,N,P),H) \cong \\
&\mathrm{Pro}-\mathrm{C.Crs}^2((L,M,N,P),L_{pC}(H)) \cong \\
&\mathrm{Crs}^2((L,M,N,P),U_{\mathrm{Crs}^2}L_{pC}(H)) \cong \\
&\mathrm{Crs}^2((L,M,N,P),LU_{C\mathfrak{C}rps}(H)) \text{ by observation} \cong \\
&\mathrm{Grps}(R(L,M,N,P),U_{\mathrm{Grps}}(H)) \cong \\
&\mathrm{Grps}(R,U_{\mathrm{Grps}}(H)) \cong \\
&\mathrm{Pro}-\mathrm{C.Grps}(P,H) \cong
\end{aligned}$$

as required; hence P≅P, independently of what L, M, N are.

In order to study conditions which imply that L, M, N and $\hat{L}$,M,N are isomorphic respectively, it is convenient to introduce a condition that we will call the "cofinality condition".

Let (L, M, N, P)

$$\begin{tikzcd}
L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\
N \arrow[r, "\mu'"] & P
\end{tikzcd}$$

be a crossed square and write $\Omega_P(L)$ for the directed subset Ω (L) the set of finite index normal subgroup of L, consisting of those $L_l \in \Omega$ (L), $L/L_l \in C$, which are P-invariant. We will say that (L, M, N, P) satisfies the cofinal condition if $\Omega_P(L)$ is cofinal in Ω (L), $\Omega_P(M)$ is cofinal in Ω (M) and $\Omega_P(N)$ is cofinal in Ω (N). Note that $\Omega_P(L) \subseteq \Omega_M(L)$ and $\Omega_P(L) \subseteq \Omega_N(L)$ so if $\Omega_P(L)$ is cofinal in Ω (L), then $\Omega_M(L)$ and $\Omega_N(L)$ are cofinal in Ω (L).

Proposition 3.8: If $P \in C$, then any crossed P-square, (L, M, N, P)

$$\begin{tikzcd}
L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\
N \arrow[r, "\mu'"] & P
\end{tikzcd}$$

satisfies the cofinality condition.

Proof: Given any $L_l \in \Omega_P(L)$, let

$$L_l' = \bigcap_{p \in P} {}^{p}L_l$$

be the intersection of all translates of L_l under the P-action. Then L_l' is P-invariant and as P in C, L_l' is of finite index $L/L_l' \in C$. As L_l' is contained in L_l so $\Omega_P(L)$ is cofinal in Ω (L). Similarly, we can show that $\Omega_P(M)$ and $\Omega_P(N)$ are cofinal in Ω (L). This comp letes the proof.

By using the method given in [1] we can give the following theorem;

Theorem 3.9: If (L, M, N, P)

$$\begin{tikzcd}
L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\
N \arrow[r, "\mu'"] & P
\end{tikzcd}$$

satisfies the cofinality condition, then $L \cong \hat{L}$, $M \cong M$, $N \cong N$.

Proof: Since $\Omega_P(M)$ is cofinal in Ω (M) by theorem above M=M. Similarly, since $\Omega_P(N)$ is cofinal in Ω (N) by theorem above N = N. On the otherhand as $\Omega_P(L)$, $\Omega_M(L)$ and $\Omega_N(L)$ are cofinal in Ω (L), $L \cong \hat{L}$.

In [1] F.J. Korkers and T. Porter showed that if crossed module ∂: M→P satisfies the cofinality condition, $\hat{\partial}: M \to P$ is a crossed module and M≅M. Hence since $\Omega_P(M)$ is cofinal in Ω (M) and $\Omega_P(N)$ is cofinal in Ω (N), M≅M and N≅N. On the other hand since $\Omega_P(L)$ is cofinal in Ω (L), $\hat{L} \cong L$.

To check the axioms we need an explicit description of $\hat{\lambda}: \hat{L} \to M$, μ: M→P, $\lambda': \hat{L} \to N$, μ′:N→P, μλ = μ′λ′: M→P and the h-map $\hat{h}: M \times N \to \hat{L}$. Given $U_{fin}P$, there is a composed homomorphism M→P→P/U. Take K_U to be its kernel then since μ is P-equivariant and P/U is finite, it follows that K_U is in $\Omega_P(M)$ and that $U \subseteq St_G(M/K_U)$. Similarly, there is a composed homomorphism $L \to M \to M/K_U$. Take T_U to be its kernel then sine λ is M-equivariant and P/U is finite, it follows that T_U is in $\Omega_M(L)$ and that $K_U \subseteq St_G(L/T_U)$. On the other hand there is also composed homomorphism L→P→P/U. Take H_U to be its kernel then since μλ is P-equivariant and P/U is finite, it follows that H_U is in Ω_P (L) and that $U \subseteq St_G(L/H_U)$. Similarly there are composed homomorphisms N→P→P/U, $L \to N \to N/K_U'$ and L→P→P/U and kernels K_U', T_U', H_U' of these morphisms respectively. It is easy to show that $H_U = T_U = T_U' = H_U'$. These observations readily imply that $\hat{\lambda}$, μ, λ', μ', $\mu\lambda = \mu'\lambda'$ and the h-map $\hat{h}$, defined by

$$\lambda_U(lT_U) = \lambda l_U K_U$$
$$\mu_U(mK_U) = \mu m_U U$$
$$\hat{\lambda}'_U(lT_U) = \lambda' l_U K_U'$$
$$\hat{\mu}'_U(nK_U') = \mu n_U U$$
$$\mu\lambda_U(lH_U) = (\mu\lambda) l_U U$$
$$\mu'\lambda'_U(lH_U) = (\mu'\lambda') l_U U$$
$$\hat{h}_U(mK_U, nK_U') = h(m,n)_U H_U$$

It is clear that

$$\mu\lambda_U = \mu_U \hat{\lambda}_U = \mu'_U \lambda'_U = \mu'\lambda'_U$$

The proof that other axioms are hold now follows from the axioms of the crossed square (L, M, N, P) and the descriptions of $\hat{\lambda}, \mu, \lambda', \mu', \mu\lambda = \mu'\lambda', \hat{h}$ and the P-action.

Corollary 3.10: If P is in C and (L, M, N, P)

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"] & P \end{tikzcd}$$

is a crossed square then $(\hat{L}, M, N, P)$

$$\begin{tikzcd} \hat{L} \arrow[r, "\hat{\lambda}"] \arrow[d, "\hat{\lambda}'"'] & \widehat{M} \arrow[d, "\hat{\mu}"] \\ \widehat{N} \arrow[r, "\widehat{\mu'}"] & \widehat{P} \end{tikzcd}$$

is a crossed square, which is the pro-C completion of (L, M, N, P)

PRO-C COMPLETION OF FREE CROSSED SQUARES

Free crossed squares

Definition 4.1: We recall from [4] the construction of free crossed square on a pair of functions (σ_2, σ_3). Let S_1, S_2 and S_3 be sets. Take F to be the free group on S_1 and suppose a function σ_2: $S_2 \to F$. Let δ: $M \to F$ be the free precrossed module on σ_2. Using the action of F on M we can form the semidirect product $M^a F$. The canonical inclusion $M \to M^a F$, $m \mapsto (m,1)$ allows us to consider M as a normal subgroup of $M^a F$. (Note that any normal inclusion is a crossed module with action giving by conjugation.) There is a second normal subgroup of $M^a F$ arising from M, namely

$$\overline{M} = \{\ (m,x) \in M^a F \ : \delta(x) = x^{-1}\ \}$$

For $m \in M$ we let m denote the element $(m^{-1}, \delta m)$ in $\overline{M}$. Suppose given a function σ_3: $S_3 \to M$ whose image lies in the kernel of the homomorphism δ: $M \to F$. There is then a corresponding function

$$\overline{\sigma_3} : S_3 \to \overline{M},\ y \mapsto (\sigma_3 y, 1).$$

We define the free crossed square on the pair of functions (σ_2, σ_3) to be the crossed square

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & \overline{M} \arrow[d, "\bar{\mu}"] \\ M \arrow[r, "\mu"] & M \ominus F \end{tikzcd}$$

where $\mu, \bar{\mu}$ are the normal inclusions and where:

- S_3 is a subset of L with σ_3 and $\bar{\sigma}_3$ the restriction of λ' and λ respectively;
- For any other crossed square of the form

$$\begin{tikzcd} K \arrow[r, "\eta"] \arrow[d, "\eta'"'] & \overline{M} \arrow[d, "\bar{\mu}"] \\ M \arrow[r, "\mu"] & M \ominus F \end{tikzcd}$$

any function τ: $S_3 \to K$ satisfying $\eta\tau = \lambda'$ extends uniquely to a morphism of crossed squares

$$\begin{pmatrix} L & \overline{M} \\ M & M^a F \end{pmatrix} \xrightarrow{\phi} \begin{pmatrix} K & \overline{M} \\ M & M^a F \end{pmatrix}$$

over the identity on M, $\overline{M}$ and $M^a F$. Clearly the free crossed square is uniquely determined, up to isomorphism, by the functions σ_2: $S_2 \to F$ and σ_3: $S_3 \to M$.

Free Pro-C crossed squares: The pro-C analogue of this concept is obtained by insisting that all crossed modules be pro-C, all maps continuous and that one replaces the sets S_1, S_2, S_3 by a profinite (i.e. compact Hausdorff totally disconnected) spaces. Explicitly we have:

Definition 4.2: Let S_1, S_2 and S_3 be profinite spaces. Take F to be the free group on S_1 and suppose a continuous function σ_2: $S_2 \to F$. Let δ: $M \to F$ be the free pro-C precrossed module on σ_2. Using the action of F on M we can form the semidirect product $M^a F$. The canonical inclusion $M \to M^a F$, $m \mapsto (m,1)$ allows us to consider M as a pro-C normal subgroup of $M^a F$. (Note that any normal inclusion is a crossed module with action giving by conjugation.) There is a second pro-C normal subgroup of $M^a F$ arising from M, namely

$$\overline{M} = \{ (m,x) \in M^a F \ : \delta(x) = x^{-1} \}$$

For $m \in M$ we let m denote the element $(m^{-1}, \delta m)$ in $\overline{M}$. Suppose given a continuous function σ_3: $S_3 \to M$ whose image lies in the kernel of the continuous homomorphism δ: $M \to F$. There is then a corresponding continuous function

$$\overline{\sigma_3} : S_3 \to \overline{M},\ y \mapsto (\sigma_3 y, 1)\,.$$

We define the free pro-C crossed square on the pair of continuous functions (σ_2, σ_3) to be the pro-C crossed square

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & \overline{M} \arrow[d, "\overline{\mu}"] \\ M \arrow[r, "\mu"] & M \ominus F \end{tikzcd}$$

where $\mu, \overline{\mu}$ are the normal inclusions and where:

- S_3 is a subset of L with σ_3 and $\overline{\sigma}_3$ the restriction of λ' and λ respectively;
- For any other pro-C crossed square of the form

$$\begin{tikzcd} K \arrow[r, "\eta"] \arrow[d, "\eta'"'] & \overline{M} \arrow[d, "\overline{\mu}"] \\ M \arrow[r, "\mu"] & M \ominus F \end{tikzcd}$$

any function τ: $S_3 \to K$ satisfying $\eta\tau = \lambda'$ extends uniquely to a morphism of pro-C crossed squares

$$\begin{pmatrix} L & \overline{M} \\ M & M^a F \end{pmatrix} \xrightarrow{\phi} \begin{pmatrix} K & \overline{M} \\ M & M^a F \end{pmatrix}$$

over the identity on M, $\overline{M}$ and M^a F. Clearly the free crossed square is uniquely determined, up to continuous isomorphism, by the functions σ_2: $S_2 \to F$ and σ_3: $S_3 \to F$.

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