The Modified Decomposition Method for the Boussinesq Equation

[1]S. Faydaoglu and [2]T. Özis

[1]Faculty of Engineering, Dokuz Eylul University, 35160, Izmir, Turkey
[2]Department of Mathematics, Science Faculty, Ege University, Izmir 35100, Turkey

Abstract: In this paper, the modified decomposition method is used to solve the generalized Boussinesq equation. This equation commonly describes the propagation of small amplitude long waves in several physical contents. The exact solution of the equation is obtained by the modified decomposition method in the form of a convergent series with easily computable components and, moreover, the convergence analysis is also given.

Key words: Nonlinear equations, Boussinesq equation, Adomian decomposition method

INTRODUCTION

Obtaining exact and explicit solutions of nonlinear partial differential equations is very important in mathematical sciences and it is one of the most inspiring and mainly active areas of the investigation. It is well-known that one large class of nonlinear partial equations belongs to the integrable partial differential equations and these equations have the infinity number of the exact solutions. The most well-known equations among them are Koteweg-de Vries equation, Sine-Gordon equation, Kawahara type equations, nonlinear Schrödinger equation, Boussinesq equations and the list can be expanded with other basic integrable equations but it is not our purpose to give all list.

In the last few decades great progress was made in the development of methods for obtaining exact solutions of nonlinear equations but the progress achieved is not adequate. Because, from our point of view, there is no single best method to obtain exact solutions of nonlinear differential equations and each method have its merits and deficiencies depending on the researchers experience and the sympathy to the method utilized. Moreover, it can be said that all these methods are problem dependant, namely some methods work well with certain problems but others not. Therefore, it is rather significant to apply some well-known methods in the literature to nonlinear partial differential equations which are not solved with that method to search possibly new exact solutions or to verify the existing solutions with different approach.

In nonlinear equations, the fourth order Boussinesq equation is a nonlinear partial differential equation that reads

$$u_{tt} = u_{xx} + u_{xxxx} + 6(u^2)_{xx}, \quad L_0 \le x \le L_1 \qquad (1)$$

with $u = u(x, t)$ is a sufficiently often differentiable function.

The initial conditions associated with the Boussinesq equation (1) are assumed to have the form

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

Boussinesq introduced Eq. (1) to describe the propagation of long waves in shallow water.

The variant of this equation also arises in other physical applications such as nonlinear lattice waves, iron sound waves in plasma and in vibrations in a nonlinear string. Moreover, it was applied to problems in the percolation of water in porous subsurface strata.

Many authors are interested in seeking soliton-like solutions, because the waveforms can change in different mechanisms and it usually has traveling wave solutions. Recently, Wazwaz [1,2] established many new traveling wave solutions for the Boussinesq and the Klein-Gordon equations by using the extended tanh method, the rational hyperbolic functions method and the rational exponential functions method to generate these new solutions. Abbasy *et al.* [3] apply the modified variational iteration method to solve a class of nonlinear partial differential equations and Boussinesq equation is used as a case-study. Wazwaz [4] uses variational iteration method is used to determine rational solutions for the KdV, the *K*(2,2), the Burgers and the cubic Boussinesq equations. Wang *et al.* [5], by means of an extended rational expansion method and symbolic computation, obtained the exact solutions of Boussinesq equation and Jimbo-Miwa equations. Hajji and Al-Khaled [6] utilized the modified Adomian

decomposition method to solve the generalized Boussinesq equation and El-Sayed, D. Kaya [7] studied the solitary-wave solutions of the (2+1)-dimensional Boussinesq equation and (3+1)-dimensional KP equation.

In this study, we extend the modified Adomian decomposition method [8] to obtain the solution of Boussinesq equation in form of Eq. (1) and give the convergence analysis of the method. The modification introduced in this study is inspired from the various works which utilize similar modifications in the literature [9-14].

THE ADOMIAN DECOMPOSITION METHOD

The principal algorithm of the Adomian decomposition method when applied to a general nonlinear equation is in the form

$$Lu + Ru + Nu = g \qquad (3)$$

The linear terms are decomposed into L+R, while the nonlinear terms are represent by Nu. L is taken as the highest order derivative to avoid difficult integration involving complicated Green's functions and R is the remainder of the linear operator L^{-1} is regarded as the inverse operator of L and is defined by a definite integration from 0 to t, i.e.,

$$L^{-1}(.) = \int_0^t \int_0^t (.) \, dt \, dt \qquad (4)$$

If L is a second-order operator, L^{-1} is a two-fold indefinite integral,

$$L^{-1}Lu = u(x,t) - u(x,0) - t \frac{\partial u(x,0)}{\partial t} \qquad (5)$$

Operating on both sides Equation (3) with L^{-1} yields

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu \qquad (6)$$

and gives

$$u(x,t) = u(x,0) + tu_t(x,0) + L^{-1}g - L^{-1}Ru - L^{-1}Nu \qquad (7)$$

The decomposition method represents the solution of Equation (7) as a series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \qquad (8)$$

The nonlinear operator, Nu, is decomposed as

$$Nu = \sum_{n=0}^{\infty} A_n \qquad (9)$$

Substituting Equations (8) and (9) onto Equation (7), we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = u_0 - L^{-1}R \sum_{n=0}^{\infty} u_n - L^{-1} \sum_{n=0}^{\infty} A_n \qquad (10)$$

where

$$u_0 = u(x,0) + tu_t(x,0) + L^{-1}g \qquad (11)$$

Consequently, it can be written as

$$\begin{aligned}
u_1 &= -L^{-1}Ru_0 - L^{-1}A_0, \\
u_2 &= -L^{-1}Ru_1 - L^{-1}A_1, \\
&\quad . \\
&\quad . \\
&\quad . \\
u_{n+1} &= -L^{-1}Ru_n - L^{-1}A_n, \quad n \geq 0,
\end{aligned} \qquad (12)$$

where A_n are Adomian's polynomials of $u_0, u_1, \ldots, u_n$ and are obtained from the formula

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[F\left(\sum_{i=0}^{\infty} \lambda^i u_i \right) \right]_{\lambda=0}, n = 0,1,2,\ldots \qquad (13)$$

Eq. (13) give

$$\begin{aligned}
A_0 &= f(u_0), \\
A_1 &= u_1 \frac{d}{du_0} f(u_0), \\
A_2 &= u_2 \frac{d}{du_0} f(u_0) + \frac{u_1^2}{2!} \frac{d^2}{du_0^2} f(u_0), \\
A_3 &= u_3 \frac{d}{du_0} f(u_0) + u_1 u_2 \frac{d^2}{du_0^2} f(u_0) + \frac{u_1^3}{3!} \frac{d^3}{du_0^3} f(u_0), \\
&\quad . \\
&\quad . \\
&\quad .
\end{aligned} \qquad (14)$$

ANALYSIS

To implement the analysis, Eq. (1) can be written in an operator form

$$L_1 u = L_x u + L_2 u + 6Nu \qquad (15)$$

where L_1, L_2 and L_x are linear differential operators and Nu is a nonlinear operator,

$$L_1 = \frac{\partial^2}{\partial t^2}, \; L_x = \frac{\partial^2}{\partial x^2}, \; L_2 = \frac{\partial^4}{\partial x^4} \qquad (16)$$

It is assumed that L^{-1} is a two-fold integral operator given by

$$L_1^{-1}(.) = \int_0^t \int_0^t (.) \, dt \, dt \qquad (17)$$

operating with the integral operator L_1^{-1} on both sides of Eq. (15) and using the given conditions we have

$$L_1^{-1} L_1 u = L_1^{-1} L_x u + L_1^{-1} L_2 u - L_1^{-1} 6Nu$$

$$u(x,t) = f(x) + tg(x) + L_1^{-1}(L_x u + L_2 u + 6Nu) \qquad (18)$$

where f and g are the functions that arise from the given initial conditions that are assumed to be prescribed. We assume that a series solution of the unknown function u (x, t) is given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \qquad (19)$$

The nonlinear term $Nu = (u^2)_{xx}$ can be decomposed into a infinite series of polynomials

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + tg(x) + L_1^{-1}(\sum_{n=0}^{\infty} u_n)_{xx}$$
$$+ L_1^{-1}(\sum_{n=0}^{\infty} u_n)_{xxxx} + L_1^{-1}(6\sum_{n=0}^{\infty} A_n) \qquad (20)$$

To determine the components $u_n(x, t)$, $n{\geq}0$, we can write the recursive relation

$$u_0(x,t) = f(x) + tg(x)$$
$$u_1(x,t) = L_1^{-1}(u_0)_{xx} + L_1^{-1}(u_0)_{xxxx} + L_1^{-1}(6A_0)$$
$$\vdots \qquad (21)$$
$$u_{n+1}(x,t) = L_1^{-1}(u_n)_{xx} + L_1^{-1}(u_n)_{xxxx} + L_1^{-1}(6A_n), \; n \geq 1$$

where A_n are the Adomian polynomials that represent the nonlinear term $(u^2)_{xx}$ and can be derived by

$$A_0 = (u_0^2)_{xx},$$
$$A_1 = 2(u_1)_{xx}(u_0)_{xx},$$
$$A_2 = (u_1^2)_{xx} + 2(u_0 u_2)_{xx}, \qquad (22)$$
$$A_3 = 2(u_1 u_2)_{xx} + 2(u_0 u_3)_{xx},$$
$$\vdots$$

Other polynomials can be generated in a like manner.

It is worth noting that the recurrence relation (21) introduces a slight variation from the original recurrence relation developed by Adomian [15]. Although this change in the formulation of the recurrence relation is slight, it introduces a qualitative tool that accelerates the convergence of the solution and minimizes the volume of calculations. The first few components of $u_n(x, t)$ follows immediately upon setting:

$$u_0(x,t) = f(x) + tg(x)$$
$$u_1(x,t) = L_1^{-1}(u_0)_{xx} + L_1^{-1}(u_0)_{xxxx} + L_1^{-1}(6A_0)$$
$$u_2(x,t) = L_1^{-1}(u_1)_{xx} + L_1^{-1}(u_1)_{xxxx} + L_1^{-1}(6A_1) \qquad (23)$$
$$u_3(x,t) = L_1^{-1}(u_2)_{xx} + L_1^{-1}(u_2)_{xxxx} + L_1^{-1}(6A_2)$$

The scheme (23) determines the components $u_n(x, t)$, $n{\geq}0$. It is, in principle, possible to calculate more components in the decomposition series to enhance the approximation. Consequently, one can recursively determine every term of the series $\sum_{n=0}^{\infty} u_n(x,t)$ and hence the solution u (x, t) is readily obtained in a series form. It is interesting to note that we obtained the series solution by using the initial conditions only. For a detailed description of Adomian decomposition method and the mo dified decomposition algorithm, we refer the reader to [15]. From Eqs. (11) and (12), we know that all the components $u_n(x, t)$ are calculable. If the series converges, the n-term partial sum

$$\phi_n = \sum_{i=0}^{n-1} u(x,t), \qquad (24)$$

will be the approximate solution since

$$\lim_{n \to \infty} \sum_{i=0}^{\infty} u_i = u \qquad (25)$$

CONVERGENCE RESULTS

The convergence results of Adomian decomposition method is first proved by Cherruault [16].

Ngarhasta *et al*. [17] and Mavoungou and Cherruault [18] have introduced a new approach for the convergence of this method. They have launched a new condition for obtaining convergence of the decomposition series to the diffusion model and Fisher's equation. In this paper, we confirm how approximate solutions of Boussinesq equation are closed to corresponding exact solutions following the steps [16, 17].

Convergence analysis: Consider a Hilbert space H, defined by

$$H = L^2((\alpha,\beta) \times [0,T])$$

the set of applications:

$$u : (\alpha,\beta) \times [0,T] \to \Re$$

with

$$\int_{(\alpha,\beta) \times [0,T]} u^2(x,s) ds d\tau < +\infty$$

and the following scalar product:

$$(u,v)_H = \int_{(\alpha,\beta) \times [0,T]} u(x,s)v(x,s) ds d\tau$$

and

$$\|u\|_H^2 = \int_{(\alpha,\beta) \times [0,T]} u^2(x,s) ds d\tau < +\infty$$

We now consider Boussinesq equation with respect to above assumptions and then the operator of nonlinear Boussinesq equation is

$$T(u) = \frac{\partial^2}{\partial t^2} = R(u) + N(u) = -\frac{\partial^2}{\partial x^2}u - \frac{\partial^4}{\partial x^4}u - 6\frac{\partial^2}{\partial t^2}u$$

where

$$L(u) = \frac{\partial^2}{\partial t^2}u, R(u) = -\frac{\partial^2}{\partial x^2}u - \frac{\partial^4}{\partial x^4}u, N(u) = -6\frac{\partial^2}{\partial x^2}u$$

The Adomian decomposition method is convergent if the following two hypotheses are satisfied [16].

(H$_1$) $(T(u) - T(v), u - v) \ge k\|u - v\|^2$, $\quad k > 0$, $\quad u, v \in H$

(H$_2$) Whatever may be M>0, there exists a constant C(M)>0 such that for u, v∈H with ‖u‖≤0, ‖v‖≤M, we have:
$(T(u) - T(v), w) \ge C(M)\|u - v\|\|w\|$ for every w∈H.

THEOREM (Sufficient Condition of Convergence for nonlinear Boussinesq equation): If N is a Lipschitzian function in H, the Adomian decomposition method applied to the following IBE

$$\frac{\partial^2}{\partial t^2}(u) = \frac{\partial^2}{\partial x^2}(u) + \frac{\partial^4}{\partial x^4}(u) + 6\frac{\partial^2}{\partial x^2}(u^2)$$

without initial and boundary conditions, converges towards a particular solution.

Proof: To prove the theorem, we will validate the conditions (H$_1$) and (H$_2$) for Boussinesq equation respectively. Now, consider the equation:

$$\frac{\partial^2}{\partial t^2}(u) = \frac{\partial^2}{\partial x^2}(u) + \frac{\partial^4}{\partial x^4}(u) + 6\frac{\partial^2}{\partial x^2}(u^2)$$

and set

$$L(u) = \frac{\partial^2}{\partial t^2}(u), R(u) = -\frac{\partial^2}{\partial x^2}(u) - \frac{\partial^4}{\partial x^4}(u),$$

$$N(u) = -6\frac{\partial^2}{\partial x^2}(u^2)$$

Therefore, we have,

$$L(u) = \frac{\partial^2}{\partial t^2}(u) = -T(u) = \frac{\partial^2}{\partial x^2}(u) + \frac{\partial^4}{\partial x^4}(u) + 6\frac{\partial^2}{\partial x^2}(u^2)$$

We assume that the convergence hypothesis (H$_1$), i.e., there exists a constant k>0, such that for u, v∈H, we have

$$(T(u) - T(v), u - v) \ge k\|u - v\|^2$$

$$T(u) - T(v) = -\frac{\partial^2}{\partial x^2}(u) - \frac{\partial^2}{\partial x^2}(v) - \frac{\partial^4}{\partial x^4}(u)$$
$$-\frac{\partial^4}{\partial x^4}(v) - 6\frac{\partial^2}{\partial x^2}(u^2) - 6\frac{\partial^2}{\partial x^2}(v^2)$$
$$= -\frac{\partial^2}{\partial x^2}(u - v) - \frac{\partial^4}{\partial x^4}(u - v) - 6\frac{\partial^2}{\partial x^2}(u^2 - v^2),$$

$$(T(u) - T(v), u - v) = \left(\begin{array}{c} -\frac{\partial^2}{\partial x^2}(u - v), u - v \\ +(-\frac{\partial^4}{\partial x^4}(u - v), u - v \end{array} \right)$$
$$+ (-6\frac{\partial^2}{\partial x^2}(u^2 - v^2), u^2 - v^2)$$

But there exists a real $\delta > 0$ such that

$$(-\frac{\partial^2}{\partial x^2}(u-v), u-v) - (\frac{\partial^4}{\partial x^4}(u-v), u-v)$$

$$= (-\frac{\partial^2}{\partial x^2}(u-v), u-v)[1 - (\frac{\partial^2}{\partial x^2}(u-v), u-v)]$$

$$\geq \delta \|u-v\|^2 (1 - \|u-v\|^2) \geq \delta \|u-v\|^2$$

hence

$$(H_1) \Rightarrow (-\frac{\partial^2}{\partial x^2}(u-v), u-v) = \delta \|u-v\|^2$$

$$(H_1) \Rightarrow [1 - (-\frac{\partial^2}{\partial x^2}(u-v), u-v)] = (1 - \|u-v\|^2)$$

Because $\dfrac{\partial^2}{\partial x^2}$ and $\dfrac{\partial^4}{\partial x^4}$ are two differential operators in H and according to the Schwartz inequality, we have

$$(-6\frac{\partial^2}{\partial x^2}(u^2-v^2), u^2-v^2) = [6(N(u)-N(v)), u^2-v^2]$$

$$\leq 6\|N(u)-N(v)\|\|u^2-v^2\|$$

As N a Lipschitzian function, we have

$$[6(N(u)-N(v)), u^2-v^2] \leq 6\alpha\|u^2-v^2\|$$

$$\Leftrightarrow -[6(N(u)-N(v)), u^2-v^2] \geq 6\alpha\|u^2-v^2\|$$

We therefore deduce

$$(T(u)-T(v), u-v) \geq \delta\|u-v\|^2 - 6\alpha\|u^2-v^2\|$$

$$\geq (\delta - 6\alpha)\|u^2-v^2\|$$

Setting $K = \delta - 6\alpha$, we obtain hypothesis (H_1) For the hypothesis (H_2), i.e., $\forall M > 0$, $\exists C(M) > 0$ such that

$$\|u\| \leq M, \quad \|v\| \leq M \Rightarrow (T(u)-T(v), w)$$

$$\leq C(M)\|u-v\|^2\|w\|, \forall w \in H \quad .$$

Indeed, we have

$$(T(u)-T(v), w) \leq 2\|u-v\|^2\|w\| + 6\alpha\|u^2-v^2\|^2\|w\|$$

$$\leq 2\|u-v\|^2\|w\| + 6\alpha\|u-v\|^2\|u+v\|^2\|w\|$$

$$\leq 2\|u-v\|^2\|w\| + 6\alpha\|u-v\|^2 4M^2\|w\|$$

$$= (2 + 24\alpha M^2)\|u-v\|^2\|w\|$$

$$= (2 + \alpha M^2)\|u-v\|^2\|w\|$$

where $C(M) = (2 + \alpha M^2)$. Thus the hypothesis (H_2) is satisfied. The proof is complete.

NUMERICAL APPLICATIONS

Example: Consider the Boussinesq equation with the following initial conditions,

$$u_{tt} = u_{xx} + u_{xxxx} + 6(u^2)_{xx} \tag{26}$$

$$u(x,0) = \alpha \operatorname{sech}^2(\frac{1}{\beta}\sqrt{\frac{\alpha}{6}}x) \tag{27}$$

$$u_t(x,0) = 2\alpha\sqrt{\frac{\alpha}{6}} \operatorname{sech}^2(\frac{1}{\beta}\sqrt{\frac{\alpha}{6}}x)\tanh(\frac{1}{\beta}\sqrt{\frac{\alpha}{6}}x) \tag{28}$$

Where α and β are arbitrary constants and

$$\beta = \sqrt{1 + \frac{2}{3}\alpha}$$

Proceeding in a usual manner, we introduce the recursive relation

$$u_0(x,t) = \alpha \operatorname{sech}^2(\frac{1}{\beta}\sqrt{\frac{\alpha}{6}}x) + 2\alpha t \operatorname{sech}^2(\frac{1}{\beta}\sqrt{\frac{\alpha}{6}}x)\tanh(\frac{1}{\beta}\sqrt{\frac{\alpha}{6}}x)$$

$$\vdots \tag{29}$$

$$u_{n+1}(x,t) = L_1^{-1}(u_n)_{xx} + L_1^{-1}(u_n)_{xxxx} + L_1^{-1}(6A_n), \quad n \geq 1$$

To make simplification, let be

$$K(x) = \cosh(\frac{1}{\beta}\sqrt{\frac{\alpha}{6}}x)$$

and

$$M(x) = \sinh(\frac{1}{\beta}\sqrt{\frac{\alpha}{6}}x)$$

and one can obtain

$$K'(x) = \frac{1}{\beta}\sqrt{\frac{\alpha}{6}}M(x),$$

$$M'(x) = \frac{1}{\beta}\sqrt{\frac{\alpha}{6}}K(x).$$

Hence;

$$u_0 = \alpha\frac{1}{K^2(x)} + \frac{\sqrt{6}\alpha^{\frac{3}{2}}}{3}\frac{M(x)}{K^3(x)}.$$

Substituting Equation (22) into Eq. (29) gives

$$u_1(x,t) = L_1^{-1}(u_0)_{xx} + L_1^{-1}(u_0)_{xxxx} + L_1^{-1}(6A_0)$$

where

$$(u_0)_x = -2\alpha \frac{K'(x)}{K^3(x)} + \frac{\sqrt{6}\alpha^{\frac{3}{2}}}{3} t \frac{M'(x)K^3(x) - 3K^2(x)K'(x)M(x)}{K^6(x)} = -\frac{\sqrt{6}\alpha^{\frac{3}{2}}}{3\beta} \frac{M(x)}{K^3(x)} + \frac{\alpha^2}{3\beta} t \frac{1}{K^2(x)} - \frac{\alpha^2}{\beta} t \frac{M^2(x)}{K^4(x)}$$

$$\frac{M^2(x)}{K^4(x)} = \frac{sh^2 a}{ch^4 a} = \frac{1}{ch^2 a} \frac{sh^2 a}{ch^2 a} = \frac{1}{ch^2 a} \frac{ch^2 a - 1}{ch^2 a} = \frac{1}{ch^2 a} (1 - \frac{1}{ch^2 a}) = \frac{1}{ch^2 a} - \frac{1}{ch^4 a} = \frac{1}{K^2(x)} - \frac{1}{K^4(x)}$$

$$(u_0)_x = -\frac{\sqrt{6}\alpha^{\frac{3}{2}}}{3\beta} \frac{M(x)}{K^3(x)} + \frac{\alpha^2}{3\beta} t \frac{1}{K^2(x)} - \frac{\alpha^2}{\beta} t \frac{1}{K^2(x)} + \frac{\alpha^2}{\beta} t \frac{1}{K^4(x)}$$

$$(u_0)_x = -\frac{\sqrt{6}\alpha^{\frac{3}{2}}}{3\beta} \frac{M(x)}{K^3(x)} - \frac{2\alpha^2}{3\beta} t \frac{1}{K^2(x)} + \frac{\alpha^2}{\beta} t \frac{1}{K^4(x)}$$

$$(u_0)_{xx} = -\frac{\sqrt{6}\alpha^{\frac{3}{2}}}{3\beta} \frac{M'(x)K^3(x) - 3K^2(x)K'(x)M(x)}{K^6(x)} + \frac{4\alpha^2}{3\beta} t \frac{K'(x)}{K^3(x)} - \frac{4\alpha^2}{\beta} t \frac{K'(x)}{K^5(x)}$$

$$= -\frac{\alpha^2}{3\beta^2} \frac{1}{K^2(x)} + \frac{\alpha^2}{\beta^2} \frac{M^2(x)}{K^4(x)} + \frac{2\sqrt{6}\alpha^{\frac{5}{2}}}{9\beta^2} t \frac{M(x)}{K^3(x)} - \frac{2\sqrt{6}\alpha^{\frac{5}{2}}}{3\beta^2} t \frac{M(x)}{K^5(x)} \Rightarrow$$

$$(u_0)_{xx} = \frac{2\alpha^2}{3\beta^2} \frac{1}{K^2(x)} - \frac{\alpha^2}{\beta^2} \frac{1}{K^4(x)} + [\frac{2\sqrt{6}\alpha^{\frac{5}{2}}}{9\beta^2} \frac{M(x)}{K^3(x)} - \frac{2\sqrt{6}\alpha^{\frac{5}{2}}}{3\beta^2} \frac{M(x)}{K^5(x)}]t$$

$$a = \frac{2\alpha^2}{3\beta^2} \frac{1}{K^2(x)} - \frac{\alpha^2}{\beta^2} \frac{1}{K^4(x)}, b = \frac{2\sqrt{6}\alpha^{\frac{5}{2}}}{9\beta^2} \frac{M(x)}{K^3(x)} - \frac{2\sqrt{6}\alpha^{\frac{5}{2}}}{3\beta^2} \frac{M(x)}{K^5(x)} \Rightarrow L_1^{-1}(u_0)_{xx} = \int_0^t \int_0^t (a+bt)dtdt = a\frac{t^2}{2} + b\frac{t^3}{6}$$

$$(u_0)_{xxx} = -\frac{4\alpha^2}{3\beta^2} \frac{K'(x)}{K^3(x)} + \frac{4\alpha^2}{\beta^2} \frac{K'(x)}{K^5(x)} + \frac{2\sqrt{6}\alpha^{\frac{5}{2}}}{9\beta^2} t \frac{M'(x)K^3(x) - 3K^2(x)K'(x)M(x)}{K^6(x)} - \frac{2\sqrt{6}\alpha^{\frac{5}{2}}}{3\beta^2} t \frac{M'(x)K^5(x) - 5K^4(x)K'(x)M(x)}{K^{10}(x)}$$

$$= -\frac{2\sqrt{6}\alpha^{\frac{5}{2}}}{9\beta^3} \frac{M(x)}{K^3(x)} + \frac{2\sqrt{6}\alpha^{\frac{5}{2}}}{3\beta^3} \frac{M(x)}{K^5(x)} + \frac{2\alpha^3}{9\beta^3} t \frac{1}{K^2(x)} - \frac{2\alpha^3}{3\beta^3} \frac{M^2(x)}{K^4(x)} - \frac{2\alpha^3}{3\beta^3} t \frac{1}{K^4(x)} + \frac{10\alpha^3}{3\beta^3} t \frac{M^2(x)}{K^6(x)} \Rightarrow$$

$$(u_0)_{xxx} = -\frac{2\sqrt{6}\alpha^{\frac{5}{2}}}{9\beta^3} \frac{M(x)}{K^3(x)} + \frac{2\sqrt{6}\alpha^{\frac{5}{2}}}{3\beta^3} \frac{M(x)}{K^5(x)} - \frac{4\alpha^3}{9\beta^3} t \frac{1}{K^2(x)} + \frac{10\alpha^3}{3\beta^3} t \frac{M^2(x)}{K^6(x)}$$

$$(u_0)_{xxxx} = -\frac{2\sqrt{6}\alpha^{\frac{5}{2}}}{9\beta^3} \frac{M'(x)K^3(x) - 3K^2(x)K(x)M(x)}{K^6(x)} + \frac{2\sqrt{6}\alpha^{\frac{5}{2}}}{3\beta^3} t \frac{M'(x)K^5(x) - 5K^4(x)K(x)M(x)}{K^{10}(x)}$$

$$+ \frac{8\alpha^3}{3\beta^3} \frac{K'(x)}{K^3(x)} + \frac{10\alpha^3}{3\beta^3} t \frac{2M(x)M'(x)K^6(x) - 6K^5(x)K'(x)M^2(x)}{K^{12}(x)}$$

$$= -\frac{2\alpha^3}{9\beta^4} \frac{1}{K^2(x)} + \frac{2\alpha^3}{3\beta^4} \frac{M^2(x)}{K^4(x)} + \frac{2\alpha^3}{3\beta^4} \frac{1}{K^4(x)} - \frac{10\alpha^3}{3\beta^4} \frac{M^2(x)}{K^6(x)} + \frac{4\sqrt{6}\alpha^{\frac{7}{2}}}{27\beta^4} t \frac{M(x)}{K^3(x)} + \frac{10\sqrt{6}\alpha^{\frac{7}{2}}}{9\beta^4} t \frac{M(x)}{K^5(x)} - \frac{10\sqrt{6}\alpha^{\frac{7}{2}}}{3\beta^4} t \frac{M^3(x)}{K^7(x)} \Rightarrow$$

$$(u_0)_{xxxx} = \frac{4\alpha^3}{9\beta^4}\frac{1}{K^2(x)} - \frac{10\alpha^3}{3\beta^4}\frac{M^2(x)}{K^6(x)} + \frac{4\sqrt{6}\alpha^{\frac{7}{2}}}{27\beta^4}t\frac{M(x)}{K^3(x)} + \frac{10\sqrt{6}\alpha^{\frac{7}{2}}}{9\beta^4}t\frac{M(x)}{K^5(x)} - \frac{10\sqrt{6}\alpha^{\frac{7}{2}}}{3\beta^4}t\frac{M^3(x)}{K^7(x)}$$

$$(u_0)_{xxxx} = \frac{4\alpha^3}{9\beta^4}\frac{1}{K^2(x)} - \frac{10\alpha^3}{3\beta^4}\frac{M^2(x)}{K^6(x)} + [\frac{4\sqrt{6}\alpha^{\frac{7}{2}}}{27\beta^4}\frac{M(x)}{K^3(x)} + \frac{10\sqrt{6}\alpha^{\frac{7}{2}}}{9\beta^4}\frac{M(x)}{K^5(x)} - \frac{10\sqrt{6}\alpha^{\frac{7}{2}}}{3\beta^4}\frac{M^3(x)}{K^7(x)}]t$$

$$c = \frac{4\alpha^3}{9\beta^4}\frac{1}{K^2(x)} - \frac{10\alpha^3}{3\beta^4}\frac{M^2(x)}{K^6(x)}, \quad d = \frac{4\sqrt{6}\alpha^{\frac{7}{2}}}{27\beta^4}\frac{M(x)}{K^3(x)} + \frac{10\sqrt{6}\alpha^{\frac{7}{2}}}{9\beta^4}\frac{M(x)}{K^5(x)} - \frac{10\sqrt{6}\alpha^{\frac{7}{2}}}{3\beta^4}\frac{M^3(x)}{K^7(x)}$$

$$L_1^{-1}(u_0)_{xxxx} = \int_0^t\int_0^t (c+dt)dtdt = c\frac{t^2}{2} + d\frac{t^3}{6}$$

$$u_0^2 = [\alpha\frac{1}{K^2(x)} + \frac{\sqrt{6}\alpha^{\frac{3}{2}}}{3}\frac{M(x)}{K^3(x)}]^2 = \alpha^2\frac{1}{K^4(x)} + \frac{2\sqrt{6}\alpha^{\frac{5}{2}}}{3}t\frac{M(x)}{K^5(x)} + \frac{2\alpha^3}{3}t^2\frac{M^2(x)}{K^6(x)}$$

$$(u_0^2)_x = -4\alpha^2\frac{K'(x)}{K^5(x)} + \frac{2\sqrt{6}\alpha^{\frac{5}{2}}}{3}t\frac{M'(x)K^5(x) - 5K^4(x)K'(x)M(x)}{K^{10}(x)} + \frac{2\alpha^3}{3}t^2\frac{2M(x)M'(x)K^6(x) - 6K^5(x)K'(x)M^2(x)}{K^{12}(x)}$$

$$(u_0^2)_x = -\frac{2\sqrt{6}\alpha^{\frac{5}{2}}}{3\beta}\frac{M(x)}{K^5(x)} - \frac{8\alpha^3}{3\beta}t\frac{1}{K^4(x)} + \frac{10\alpha^3}{3\beta}t\frac{1}{K^6(x)} - \frac{4\sqrt{6}\alpha^{\frac{7}{2}}}{9\beta}t^2\frac{M(x)}{K^5(x)} + \frac{2\sqrt{6}\alpha^{\frac{7}{2}}}{3\beta}t^2\frac{M(x)}{K^7(x)}$$

$$(u_0^2)_{xx} = -\frac{2\sqrt{6}\alpha^{\frac{5}{2}}}{3\beta}\frac{M(x)K^5(x) - 5K^4(x)K'(x)M(x)}{K^{10}(x)} - \frac{32\alpha^3}{3\beta}t\frac{K'(x)}{K^5(x)} - \frac{20\alpha^3}{\beta}t\frac{K'(x)}{K^7(x)}$$

$$- \frac{4\sqrt{6}\alpha^{\frac{7}{2}}}{9\beta}t^2\frac{M'(x)K^5(x) - 5K^4(x)K'(x)M(x)}{K^{10}(x)} + \frac{2\sqrt{6}\alpha^{\frac{7}{2}}}{3\beta}t^2\frac{M'(x)K^7(x) - 7K^6(x)K'(x)M(x)}{K^{14}(x)}$$

$$(u_0^2)_{xx} = -\frac{2\alpha^3}{3\beta^2}\frac{1}{K^4(x)} + \frac{10\alpha^3}{3\beta^2}\frac{M^2(x)}{K^6(x)} + \frac{16\sqrt{6}\alpha^{\frac{7}{2}}}{9\beta^2}t\frac{M(x)}{K^5(x)} - \frac{10\sqrt{6}\alpha^{\frac{7}{2}}}{3\beta^2}t\frac{M(x)}{K^7(x)} - \frac{4\alpha^4}{9\beta^2}t^2\frac{1}{K^4(x)}$$

$$+ \frac{20\alpha^4}{9\beta^2}t^2\frac{M^2(x)}{K^6(x)} + \frac{2\alpha^4}{3\beta^2}t^2\frac{1}{K^6(x)} - \frac{14\alpha^4}{3\beta^2}t^2\frac{M^2(x)}{K^8(x)}$$

$$(u_0^2)_{xx} = -\frac{2\alpha^3}{3\beta^2}\frac{1}{K^4(x)} + \frac{10\alpha^3}{3\beta^2}\frac{M^2(x)}{K^6(x)} + [-\frac{14\sqrt{6}\alpha^{\frac{7}{2}}}{9\beta^2}\frac{M(x)}{K^5(x)} + \frac{10\sqrt{6}\alpha^{\frac{7}{2}}}{3\beta^2}\frac{M^3(x)}{K^7(x)}]t$$

$$+ [-\frac{4\alpha^4}{9\beta^2}\frac{1}{K^4(x)} + \frac{20\alpha^4}{9\beta^2}\frac{M^2(x)}{K^6(x)} + \frac{2\alpha^4}{3\beta^2}\frac{1}{K^6(x)} - \frac{14\alpha^4}{3\beta^2}\frac{M^2(x)}{K^8(x)}]t^2$$

$$e = -\frac{2\alpha^3}{3\beta^2}\frac{1}{K^4(x)} + \frac{10\alpha^3}{3\beta^2}\frac{M^2(x)}{K^6(x)}, \quad f = -\frac{14\sqrt{6}\alpha^{\frac{7}{2}}}{9\beta^2}\frac{M(x)}{K^5(x)} + \frac{10\sqrt{6}\alpha^{\frac{7}{2}}}{3\beta^2}\frac{M^3(x)}{K^7(x)}$$

$$g = -\frac{4\alpha^4}{9\beta^2}\frac{1}{K^4(x)} + \frac{20\alpha^4}{9\beta^2}\frac{M^2(x)}{K^6(x)} + \frac{2\alpha^4}{3\beta^2}\frac{1}{K^6(x)} - \frac{14\alpha^4}{3\beta^2}\frac{M^2(x)}{K^8(x)}$$

$$L_1^{-1}(u_0^2)_{xx} = \int_0^t\int_0^t(e + ft + gt^2)dtdt = e\frac{t^2}{2} + f\frac{t^3}{6} + g\frac{t^4}{12}$$

$$u_1(x,t) = L_1^{-1}(u_0)_{xx} + L_1^{-1}(u_0)_{xxxx} + L_1^{-1}(6A_0)$$

$$a = \frac{2\alpha^2}{3\beta^2}\frac{1}{K^2(x)} - \frac{\alpha^2}{\beta^2}\frac{1}{K^4(x)}, \ b = \frac{2\sqrt{6}\alpha^{\frac{5}{2}}}{9\beta^2}\frac{M(x)}{K^3(x)} - \frac{2\sqrt{6}\alpha^{\frac{5}{2}}}{3\beta^2}\frac{M(x)}{K^5(x)} \Rightarrow L_1^{-1}(u_0)_{xx} = \int_0^t\int_0^t(a + bt)dtdt = a\frac{t^2}{2} + b\frac{t^3}{6}$$

$$c = \frac{4\alpha^3}{9\beta^4}\frac{1}{K^2(x)} - \frac{10\alpha^3}{3\beta^4}\frac{M^2(x)}{K^6(x)}, \ d = \frac{4\sqrt{6}\alpha^{\frac{7}{2}}}{27\beta^4}\frac{M(x)}{K^3(x)} + \frac{10\sqrt{6}\alpha^{\frac{7}{2}}}{9\beta^4}\frac{M(x)}{K^5(x)} - \frac{10\sqrt{6}\alpha^{\frac{7}{2}}}{3\beta^4}\frac{M^3(x)}{K^7(x)}$$

$$L_1^{-1}(u_0)_{xxxx} = \int_0^t\int_0^t(c + dt)dtdt = c\frac{t^2}{2} + d\frac{t^3}{6}$$

$$e = -\frac{2\alpha^3}{3\beta^2}\frac{1}{K^4(x)} + \frac{10\alpha^3}{3\beta^2}\frac{M^2(x)}{K^6(x)}, \quad f = -\frac{14\sqrt{6}\alpha^{\frac{7}{2}}}{9\beta^2}\frac{M(x)}{K^5(x)} + \frac{10\sqrt{6}\alpha^{\frac{7}{2}}}{3\beta^2}\frac{M^3(x)}{K^7(x)}$$

$$g = -\frac{4\alpha^4}{9\beta^2}\frac{1}{K^4(x)} + \frac{20\alpha^4}{9\beta^2}\frac{M^2(x)}{K^6(x)} + \frac{2\alpha^4}{3\beta^2}\frac{1}{K^6(x)} - \frac{14\alpha^4}{3\beta^2}\frac{M^2(x)}{K^8(x)}$$

$$L_1^{-1}(u_0^2)_{xx} = \int_0^t\int_0^t(e + ft + gt^2)dtdt = e\frac{t^2}{2} + f\frac{t^3}{6} + g\frac{t^4}{12}$$

$$u_1 = a\frac{t^2}{2} + b\frac{t^3}{6} + c\frac{t^2}{2} + d\frac{t^3}{6} + 6(e\frac{t^2}{2} + f\frac{t^3}{6} + g\frac{t^4}{12})$$

$$u_1 = (a + c + 6e)\frac{t^2}{2} + (b + d + 6f)\frac{t^3}{6} + 6g\frac{t^4}{12} = -\frac{1}{162}t^2[-81a - 81c - 486e + (-27b - 27d - 162f)t - 81gt^2]$$

$$u_1 = -\frac{1}{162}\frac{t^2}{K^8(x)\beta^4}\{-54\alpha^2\beta^2K^6(x) + 81\alpha^2\beta^2K^4(x) - 36\alpha^3K^6(x) + 270\alpha^3K^2(x)M^2(x) + 324\alpha^3\beta^2K^4(x)$$

$$-1620\alpha^3\beta^2K^2(x)M^2(x) + [-6\sqrt{6}\alpha^{\frac{5}{2}}\beta^2K^5(x)M(x) + 18\sqrt{6}\alpha^{\frac{5}{2}}\beta^2K^3(x)M(x) - 4\sqrt{6}\alpha^{\frac{7}{2}}K^5(x)M(x) - 30\sqrt{6}\alpha^{\frac{7}{2}}K^3(x)M(x)$$

$$-90\sqrt{6}\alpha^{\frac{7}{2}}K(x)M^3(x) + 252\alpha^{\frac{7}{2}}\beta^2K^3(x)M(x) - 540\sqrt{6}\alpha^{\frac{7}{2}}\beta^2K(x)M^3(x)]t + [36\alpha^4\beta^2K^4(x) - 180\alpha^4\beta^2K^2(x)M^2(x)$$

$$-54\alpha^4\beta^2K^2(x) + 378\alpha^4\beta^2M^2(x)]t^2\}$$

Therefore;

$$u_0 = \alpha\frac{1}{K^2(x)} + 2\alpha t\sqrt{\frac{\alpha}{6}}\frac{M(x)}{K^3(x)}$$

$$u_1 = -\frac{1}{162}t^2\left(\frac{N(x)+H(x)+R(x)+P(x)}{K^8(x)\beta^4}\right)$$

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where

$$K(x) = \cosh(\frac{1}{\beta}\sqrt{\frac{\alpha}{6}}x), M(x) = \sinh(\frac{1}{\beta}\sqrt{\frac{\alpha}{6}}x)$$

$$N(x) = 270\alpha^3 K^2(x)M^2(x) + 81\alpha^2\beta^2 K^4(x) + 18\sqrt{6}\alpha^{\frac{5}{2}}\beta^2 tK^3(x)M(x)a$$

$$H(x) = -180\alpha^4\beta^2 t^2 K^2(x)M^2(x) + 36\alpha^4\beta^2 t^2 K^4(x) - 90\sqrt{6}\alpha^{\frac{7}{2}}tK(x)M^3(x) + 378\alpha^4\beta^2 t^2 M^2(x)$$

$$R(x) = -54\alpha^4\beta^2 t^2 K^2(x) - 54\alpha^2\beta^2 K^6(x) + 324\alpha^3\beta^2 K^4(x) - 4\sqrt{6}\alpha^{\frac{7}{2}}tK^5(x)M(x) + 252\alpha^{\frac{7}{2}}\beta^2 tK^3(x)M(x)$$

$$P(x) = -6\sqrt{6}\alpha^{\frac{5}{2}}\beta^2 tK^5(x)M(x) - 30\sqrt{6}\alpha^{\frac{7}{2}}tK^3(x)M(x) - 540\sqrt{6}\alpha^{\frac{7}{2}}\beta^2 tK(x)M^3(x)$$
$$-1620\alpha^3\beta^2 K^2(x)M^2(x) - 36\alpha^3 K^6(x)$$

and from Eq.(19) the first order approximate solution reads

$$u(x,t) = u_0 + u_1 + ...$$
$$= \alpha\frac{1}{K^2(x)} + 2\alpha t\sqrt{\frac{\alpha}{6}}\frac{M(x)}{K^3(x)} - \frac{1}{162}t^2\left(\frac{N(x)+H(x)+R(x)+P(x)}{K^8(x)\beta^4}\right) + ...$$

which is an convergent series according to the theorem proved for the convergence for nonlinear Boussinesq equation.

CONCLUSION

In this study, the modified decomposition method is used to solve the generalized Boussinesq equation. The method provides the exact solution in the absence of round off error. It has also been shown that the method is theoretically convergent and detailed convergence analysis is given. The most interesting future of the modification is, unlike classical decomposition method, accelerates the convergence of the solution and minimizes the volume of calculations This is done by choosing the first term of the Adomian series, u_0, as a function of t and x which verifies the initial and boundary conditions and by introducing a slight variation of recurrence relation from the original recurrence relation developed by Adomian. Future work includes generalization of the technique to solve other class of nonlinear differential equations of mathematical physics.

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