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A Linear Approach to Node-restricted Minmax Regret Robust 1-median on a Tree

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Abstract: In this paper, we consider the node-restricted minmax regret 1-median problem on a tree network where node weights are deterministic and edge lengths are subject to uncertainty with unknown symmetric distribution. There have been proposed some complicated and time-consuming algorithms to obtain robust solutions for robust 1-median problem. We formulate the robust regret function as a mixed integer problem and extend the linear counterpart such that the complexity of our proposed model is the same as the traditional 1-median problem. We use a numerical example to demonstrate the performance of our proposed method.

Key words: Minmax regret approach . robust optimization . networks . linearization

INTRODUCTION

During the past few decades, 1-median problem has been used among practitioners for different applications as an efficient tool to reduce transportation cost. A typical 1-median problem is demonstrated as a tree T = (V, E) with node set V and edge set E. Let P(x, y) be the unique path connecting two points x and y on the tree and d (x, y) be the length of the path. For any node $v_i \in V$, a nonnegative deterministic weight w_i is given. The 1-median of the tree is a point x on the tree such that the total weighted distance from x to all nodes, $\sum_{v_i \in V} w_i d(x, v_i)$, is minimized. When the facilities are relocated, the minmax regret approach becomes a promising tool to reduce transportation cost. The minmax regret approach was first applied to a robust location problem by Kouvelis *et al*. [1] where the uncertainty on the input parameters were handled using the advances on robust optimization. Averbakh and Berman [2] extended the 1-median robust problem for network applications. Chen and Lin [3] developed an algorithm to reduce the burden of the computation of 1-median problem. Goldman [4] studied an especial form of 1-median problem where center location on a general network was located. One of the primary assumptions on classical 1-median problem is to consider all input parameters in deterministic form. However, this simple assumption has been argued by many since the input parameters may always suffer from the existing noise. Therefore, we need to develop a method to assure us that a small perturbation on the data does not change the feasibility of the optimal solution. There are different methods to handle the uncertainty in the data which lead us to have more complicated problem formulation. The proposed robust 1-median problem formulation of this paper has the capability to handle the uncertainty without changing the structure of the original mixed integer problem. The proposed robust optimization developed by Bertsimas and Thiele [5] uses a linear counterpart as part of the original problem. We use the linear counterpart as part of our robust 1-median problem to find the maximum regret for all nodes in all possible scenarios. This paper is organized as follows. We first present the problem statement and the necessary notations and the solution procedure and the implementation of the proposed method is illustrated using a numerical example. Finally, the conclusion remarks are given at the end to summarize the contribution of the paper.

MINMAX-REGRET 1-MEDIAN APPROACH

One of the primary assumptions on many 1-median problems is to have all input parameters in deterministic form. This simple assumption may often make our final optimal solution infeasible. Ben-Tal and Nemirovski [6] used some benchmark problems from NETLIB library and showed that a small perturbation on input data could make the final solutions useless since most of them were not even feasible. They also applied their robust method on the perturbed problems. While the optimal solutions of the robust problems were almost the same as the original ones, the final solutions remained immune against the perturbation on the data. The uncertainty in 1-median problem may be considered on the distance among various nodes which

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corresponds to uncertainty in transportation times or travel costs along the edges. Let u, $\nu \in V$, $[u,v] \in E$ and d (u,v) be the corresponding edge length. For edge $[u,v] \in E$, we assume that d (u,v) is random with unknown distribution and can take any value in the interval $[\underline{d}(u,v), \overline{d}(u,v)]$. Values $\underline{d}(u,v)$ and $\overline{d}(u,v)$ represent the upper and the lower bounds on the $[u,v] \in E$ length and the interval $[\underline{d}(u,v), \overline{d}(u,v)]$ is called the interval of uncertainty for the length of edge $[u,v] \in E$. Let l be length-scenario for each feasible realization of edge lengths and L be the set of all feasible scenarios. A feasible lengths-scenario $l \in L$ happens if $\underline{d}(u,v) \le d(u,v) \le \overline{d}(u,v)$ for all $[u,v] \in E$. We denote m^l as the scenario median for tree T under scenario $l \in L$ with $F^*(l)$ as the optimal objective value. A scenario median is an 1-median for some scenario $l \in L$. For a point $x \in T$, value $F(l,x) - F^*(l)$ is called the regret for location x under scenario l denoted by $R^l(x)$. The worst-case regret of x with respect to m^l is given by the following problem:

$$\mathrm{MAXREG}(x) = \max_{l \in L}(F(l,x) - F^*(l)) \tag{1}$$

If $d^l(x,\nu_i)$ shows the length of unique path $P(x,\nu_i)$ that connects x and ν_i under scenario l, then F(l,x) can be given by:

$$F(l,x) = \sum_{\nu_i \in V} w_i d^l(x, \nu_i) \tag{2}$$

Therefore, we can show the optimization problem (1) as

$$\mathrm{MAXREG}(x) = \max_{l \in L} \sum_{\nu_i \in V} w_i d^l(x, \nu_i) - \sum_{\nu_i \in V} w_i d^l(m^l, \nu_i) \tag{3}$$

Clearly, F (l,x), $R^l(x)$ are convex functions on tree T. The definition of convexity on tree and related properties can be found in work by Tansel *et al*. [7]. The minmax regret 1-median x^* is defined as a point with the minimum value of optimization problem (3) where it has the least maximum regret compared with the best location of each scenario. When the minmax regret 1-median is restricted to be a node, it is called the node-restricted minmax-regret 1-median denoted by ν^*. In next section, we first propose a stochastic version of the optimization problem (3) and then propose a linearization approach to solve the resulted problem.

STOCHASTIC MINMAX-REGRET 1-MEDIAN PROBLEM

Given a node $\nu \in V$, we are interested in the worst-case scenario l which gives the maximum regret of ν among all scenarios L. For edge $[u,v] \in E$, we assume that $\tilde{d}(u,v)$ is random variable with unknown distribution and can take any value in the interval $[\underline{d}(u,v), \overline{d}(u,v)]$. In other words, $\tilde{d}(u,v)$ shows any realization of $[u,v] \in E$ length. Therefore we have:

$$\mathrm{MAXREG}(x) = \max \sum_{\nu_i \in V} w_i \tilde{d}(\nu, \nu_i) - \sum_{\nu_i \in V} w_i \tilde{d}(\nu^*, \nu_i) \tag{4}$$

In problem (4), we are interested in determining the node ν^* which obtains the maximum regret of node ν among any realization of edge lengths in tree T. The optimization problem (4) is called as a robust version of the maximum regret of a node in which the edge lengths of tree T are assumed to belong to an interval of uncertainty. In robust optimization problems, uncertain parameters may be modeled as either discrete or continuous. Discrete parameters are modeled using the scenario approach and continuous parameters are often assumed to lie in some prespecified intervals. The benefits and the disadvantages of each approach (scenario approach and prespecified intervals) as well as a comprehensive survey of location models under uncertainty are presented by Snyder [8]. The idea of robust optimization on traditional Linear Programming (LP) problems was first introduced by Soyster [9] who proposed a worst case model for LP optimization. Since then, many people look for less conservative methods of robust optimization and the least conservative methods were proposed by Ben-Tal and Nemirovski [10] and Goldfarb and Iyangar [11]. Bertsimas and Sim [12, 13] developed a new robust optimization where the structure of the robust formulation remains the same as the original one. The robust counterparts of the nominal problems generally are in the form of conic quadratic problems and even linear optimization problems of slightly larger size (see Bertsimas and Sim, 2004b). The proposed 1-median problem of this paper is formulated using the robust method introduced by Bertsimas and Thiele [5]. Let $\tilde{d}(\nu,\nu_i)$ be the scaled deviation of distance from its lower value. Therefore we have

$$z_{ij} = (\tilde{d}(\nu,\nu_i) - \underline{d}(\nu,\nu_i)) / (\overline{d}(\nu,\nu_i) - \underline{d}(\nu,\nu_i))$$

The scaled deviation takes value in [0, 1]. Moreover, we impose a budget of uncertainty in the following sense: The total scaled variation of the uncertain parameters (here are the edge lengths) cannot exceed some threshold Γ which is not necessarily integer:

$$\sum_{E(\nu,\nu_i) \in T} |z_i| \le \Gamma \tag{5}$$

The value of Γ can change from zero, all edge lengths take their lower values, to T, where they take their upper values. Obviously, by taking $\Gamma = T$, we can obtain the worst case which is the optimal solution for the optimization problem (4). In other words, the uncertainty budget (Γ), is an adjusting parameter which adjusts the robustness of solution with the degree decision maker's conservativeness. Before defining the stochastic model, consider the following set,

$$D = \left\{ d \,\middle|\, \begin{array}{l} \tilde{d}(v, v_i) \in \left[\underline{d}(v, v_i), \bar{d}(v, v_i)\right] \ \forall i \in V, \\ \displaystyle\sum_{i \in V} \frac{\tilde{d}(v, v_i) - \underline{d}(v, v_i)}{\bar{d}(v, v_i) - \underline{d}(v, v_i)} \le \Gamma \end{array} \right\} \tag{6}$$

Therefore, the proposed robust optimization problem can be formulated as follows:

$$\begin{aligned} \max & \sum_{v_i \in V} w_i \tilde{d}(v, v_i) - \sum_{v_i, v_j \in V} w_i \tilde{d}(v_j, v_i) y_j \\ \text{s.t.} & \sum_{v_j \in V} y_j = 1, \ \ y_j \in \{0,1\}, \ \ \tilde{d}(v, v_i) \in D, \ \ v_i \in V \end{aligned} \tag{7}$$

In model (7), the uncertainty happens in variable's coefficients in the objective function and it shows the uncertainty in edge lengths for our case. Also, y_j is a binary variable which takes 1 if the optimal median solution under uncertainty falls in node v_j and zero otherwise. Next section, we propose a linearization approach to obtain the robust linear counterpart for the proposed optimization problem.

THE ROBUST MINMAX REGRET COUNTERPART

In this section, we propose the robust minmax regret linear counterpart for robust minmax-regret problem presented in the previous section. Bertsimas and Thiele [5] presented a robust counterpart which is of the same class as the nominal problem. It is also shown that the robust counterpart of a mixed-integer programming problem is itself another mixed-integer programming problem. The most attractive features of this approach are the possibility of solving the robust counterpart by standard optimization packages and also considering the level of conservativeness of the solution. The stochastic regret problem (7) has the following robust linear counterpart:

$$\begin{aligned} \max \ & R_v \\ \text{s.t.} \ & \sum_{v_i \in V} w_i d(v, v_i) - \sum_{v_i, v_j \in V} w_i d(v_j, v_i) y_j + q\Gamma + \sum_{E(v_i, v_j) \in J} r_{ij} = R_v \\ & q + r_{ij} \le \hat{d}(v_i, v_j) . u_{ij}, \quad y_i \le u_i \\ & \sum_{v_j \in V} y_j = 1, \ \ y_j \in \{0,1\}, \ \ q \ge 0, \ \ r_{ij} \ge 0, \ \ u_{ij} \ge 0 \end{aligned} \tag{8}$$

In model (8), $\hat{d}(v_i, v_j) = \bar{d}(v_i, v_j) - \underline{d}(v_i, v_j)$ is the interval length of edge E (v_i, v_j)$\in$T. The role of uncertainty budget (Γ), is to provide the trade-off between robustness and the performance of the solutions. As we are interested in the worst case scenario to obtain the maximum regret for each node (R_v), the upper value of uncertainty budget, $\Gamma = T$, must be considered in optimization problem (8). If J shows the set of all uncertain edges in tree T, then we place $\Gamma = |J|$ in (8). To show the performance of our proposed model, let us present a simple but illustrative 1-median problem with 3 nodes.

ILLUSTRATIVE EXAMPLE

Consider the 3-vertex network given in Fig. 1, where 6.5, 3 and 5 are the demands to be covered. All node weights are deterministic and all edge lengths are uncertain and may take any value within the given intervals.

In node-restricted minmax regret approach, we first calculate the regret for all nodes for all feasible scenarios and then choose the node with the smallest maximum regret. To show the performance of the model presented in this paper, let us compare the results in Table 1 to what is achieved through our proposed model. In Table 1, we calculate the regret of each node in any feasible scenario. Note that we list only those scenarios in which the edge lengths can take their upper values.

The minmax-regret 1-median is the node with the smallest maximum regret that is v_3 with the maximum regret 6 in Table 1. If the optimization problem (8) is solved for each node, the maximum regret 13, 40 and 6 will be achieved for v_1, v_2 and v_3, respectively.

CONCLUSIONS AND FUTURE RESEARCH DIRECTIONS

We have presented a new robust 1-median problem where the edge lengths are subject to

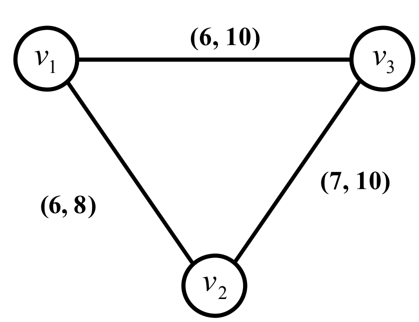

Fig. 1: A 3-node network with uncertain edge lengths

Table 1: The maximum scenario regret of each node

Scenario	ν^*	$F(\nu^*)$	$F(l,\nu_1)$	R_{ν_1}	$F(l,\nu_2)$	R_{ν_2}	$F(l,\nu_3)$	R_{ν_3}
$\bar{d}_{12}$	ν_3	51	60	9	82	31	51	0
$\bar{d}_{13}$	ν_3	71	78	7	72	1	71	0
$\bar{d}_{23}$	ν_1	54	54	0	90	36	60	6
$\bar{d}_{12}, \bar{d}_{13}$	ν_3	71	84	13	82	11	71	0
$\bar{d}_{12}, \bar{d}_{23}$	ν_1, ν_3	60	60	0	100	40	60	0
$\bar{d}_{13}, \bar{d}_{23}$	ν_1	78	78	0	90	12	80	2
$\bar{d}_{12}, \bar{d}_{13}, \bar{d}_{23}$	ν_3	80		4	100	20	80	0
MAXREG $(v_i) = \max R_{v_i}$				13		40		6

uncertainty. The proposed method of this paper uses the recent advances of robust optimization techniques to handle the uncertainty with input parameters. The proposed robust 1-median optimization has the same mixed integer structure. Therefore, one may use a direct optimization method to solve the resulted problem. This work can be extended in different forms. Since the proposed method of this paper is classified as mixed integer problem, a direct implementation of the branch and bound is not possible for large-scale problems. Therefore, one may use some well known meta-heuristics such as genetic algorithm, ant colony, etc to find the near optimal solutions. The other interesting research area is to consider uncertainty in all input parameters such as demand and we leave it for interested researchers as future work.

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