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Characterizations of Discrete Distributions Based on Factorial Cumulants

[1]Masood Anwar and [2]Munir Ahmad

[1]Department of Mathematics, COMSATS Institute of Information Technology,
Park Road Chak Shahzad, 44000, Islamabad, Pakistan
[2]National College of Business Administration and Economics, Lahore, Pakistan

Abstract: In this paper, characterizations of the binomial, negative binomial, geometric, Poisson binomial, Hermite and Poisson distributions are done through their factorial cumulants.

Key words: Characterization . discrete distributions . factorial cumulants

INTRODUCTION

Summarization of statistical data without loosing information is one of the fundamental objectives of statistical data analysis i.e. to choose an adequate model to describe the observed values obtained in an experiment. For this purpose the characterization theorems of distributions can be useful. They form an essential tool of statistical inference. For a detailed account of history of the theory of characterizations see [1, 2, 3, 4]. The survey by [5] covers a substantial number of results in the field.

[6, 7] prove that equality of the mean and variance characterizes the Poisson distribution among the *power series distributions* (PSD). [8] extends the PSD to multivariate distributions and shows that knowledge of first two moments (or, equivalently, the first two factorial moments; cumulants, or factorial cumulants) as functions of a parameter is sufficient to determine the distribution. [9] gives the mean-variance result by proving that within PSD, μ_2 = m (1-mc) if and only if *X* has a binomial, Poisson, or negative binomial distribution according to whether *c* is a positive integer, zero, or negative integer, respectively. [10] obtain characterizations for the binomial, Grassia I-binomial, (carrier-borne epidemic) and randomized occupancy distributions via their factorial moments.

We in this paper have characterized the binomial, negative binomial, geometric, Poisson-binomial, Hermite and Poisson distributions via their factorial cumulants.

CHARACTERIZATIONS BASED ON FACTORIAL CUMULANTS

Theorem 2.1: Let

$$G(z)=\sum_{x=0}^{n} z^{x}P(X=x)$$

be the probability generating function (*pgf*) of a distribution with support $0,1,2,\ldots,n$, $n\in\mathbb{Z}^{+}$ and parameter $p, 0<p<1$ such that $\lim_{p\to 0}\ln G(z)=0$. Then

$$\frac{d\kappa_{[r]}}{dp}=(-1)^{r+1}n\,r!\,p^{r-1},\quad q=1-p,\ 1\le r\le n \qquad (2.1)$$

if and only if *X* has binomial distribution with probability mass function (*pmf*)

$$P(X=x)=\binom{n}{x}p^{x}q^{n-x},\ q=1-p \qquad (2.2)$$

Proof: Suppose *X* has binomial distribution with *pmf* (2.2), then by definition

$$G(1+t)=\sum_{x=0}^{n}\frac{n!}{x!(n-x)!}\left((1+t)p\right)^{x}q^{n-x}=\left(1+pt\right)^{n}$$

$$\ln G(1+t)=n\ln(1+pt),$$

$$=n\left[pt-\frac{(pt)^{2}}{2}+\frac{(pt)^{3}}{3}-\cdots\right]=\sum_{r=1}^{\infty}(-1)^{r+1}n(r-1)!p^{r}\frac{t^{r}}{r!},$$

the coefficient of $t^r/r!$ is

$$\kappa_{[r]}=(-1)^{r+1}n(r-1)!p^{r},\ 1\le r\le n,$$

and after differentiating w.r.t 'p' we get (2.1).

Suppose (2.1) holds and after integration we have

$$\kappa_{[r]}=(-1)^{r+1}n(r-1)!p^{r}+C_{r},\ 1\le r\le n,$$

Corresponding Author: Dr. Masood Anwar Department of Mathematics, COMSATS Institute of Information Technology, Park Road Chak Shahzad, 44000, Islamabad, Pakistan

where C_r is a constant of integration.

Since $\lim_{p\to 0} \ln G(z)=0$, the limiting factorial cumulant generating function (*fcgf*) is

$$\lim_{p\to 0}\sum_{r=1}^{\infty}\left((-1)^{r+1}n(r-1)!p^r+C_r\right)\frac{t^r}{r!}=0,$$

which implies that $C_r = 0$, $1\le r\le n$ Hence the *fcgf* is

$$\ln G(1+t)=n\sum_{r=1}^{\infty}(-1)^{r+1}\frac{(pt)^r}{r},\quad =n\ln(1+pt),$$

$$G(1+t)=(1+pt)^n,\ G(z)=(q+pz)^n,$$

$$P(X=x)=\left(\frac{1}{x!}\frac{d^xG(z)}{dz^x}\right)_{z=0}$$

and we get (2.2).

Theorem 2.2: Let

$$G(z)=\sum_{x=0}^{\infty}z^xP(X=x)$$

be the probability generating function (*pgf*) of a distribution with support 0, 1, 2,… and parameters k, k>0 and p, 0<p<1, such that $\lim_{p\to 1}\ln G(z)=0$. Then

$$\frac{d\kappa_{[r]}}{dp}=-kr!\left(\frac{q}{p}\right)^r\frac{1}{pq},\ q=1-p,\ r\ge 1 \qquad (2.3)$$

if and only if *X* has negative binomial distribution with *pmf*

$$P(X=x)=\binom{x+k-1}{x}p^kq^x,\ q=1-p\ \ x=0,1,2,\cdots \quad (2.4)$$

Proof: Suppose *X* has negative binomial distribution with *pmf* (2.4), then by definition

$$G(1+t)=p^k\sum_{x=0}^{\infty}\binom{x+k-1}{x}((1+t)q)^x=p^k(1-q(1+t))^{-k}$$

$$=\left(1-\frac{q}{p}t\right)^{-k}$$

$$\ln G(1+t)=-k\ln\left(1-\frac{q}{p}t\right)$$

$$=k\left[\frac{q}{p}t+\left(\frac{q}{p}\right)^2\frac{t^2}{2!}+2\left(\frac{q}{p}\right)^3\frac{t^3}{3!}+\cdots+(r-1)!\left(\frac{q}{p}\right)^r\frac{t^r}{r!}+\cdots\right]$$

the coefficient of $t^r/r!$ is

$$\kappa_{[r]}=k(r-1)!\left(\frac{q}{p}\right)^r,\ r\ge 1,$$

differentiating w.r.t 'p' we get (2.3).

Suppose (2.3) holds and after integration we have

$$\kappa_{[r]}=k(r-1)!\left(\frac{q}{p}\right)^r+C_r,\ r\ge 1,$$

where C_r is a constant of integration.

Since $\lim_{p\to 1}\ln G(z)=0$, the limiting *fcgf* is

$$\lim_{p\to 1}\sum_{r=1}^{\infty}\left(k(r-1)!\left(\frac{q}{p}\right)^r+C_r\right)\frac{t^r}{r!}=0,$$

which implies that $C_r = 0$, $r\ge 1$. Hence the *fcgf* is

$$\ln G(1+t)=k\sum_{r=1}^{\infty}(r-1)!\left(\frac{q}{p}\right)^r\frac{t^r}{r!}=-k\ln\left(1-\frac{qt}{p}\right),$$

$$G(1+t)=\left(1-\frac{qt}{p}\right)^{-k}$$

$$G(z)=\left(\frac{1-qz}{p}\right)^{-k}$$

$$P(X=x)=\left(\frac{1}{x!}\frac{d^xG(z)}{dz^x}\right)_{z=0}$$

and we get (2.4).

Corollary 2.1: Let

$$G(z)=\sum_{x=0}^{\infty}z^xP(X=x)$$

be the probability generating function (*pgf*) of a distribution with support 0, 1, 2,… and parameter p, 0<p<1, such that $\lim_{p\to 1}\ln G(z)=0$. Then

$$\frac{d\kappa_{[r]}}{dp}=-r!\left(\frac{q}{p}\right)^r\frac{1}{pq},\quad r\ge 1 \qquad (2.5)$$

if and only if X has geometric distribution with *pmf*

$$P(X = x) = p q^{x}, \quad x = 0,1,2,\cdots \tag{2.6}$$

Proof: Since geometric distribution is a special case of negative binomial distribution at k = 1, therefore if we put k = 1 in Theorem 2.2 we get Corollary 2.1.

Theorem 2.3: Let

$$G(z) = \sum_{x=0}^{nm} z^{x} P(X = x)$$

be the probability generating function (*pgf*) of a distribution with support 0, 1, 2,…,nm, n, m$\in \mathbb{Z}^{+}$ and parameters λ, λ>0 and p, 0<p<1 such that $\lim_{p\to 0} \ln G(z) = 0$. Then

$$\frac{d\kappa_{[r]}}{dp} = \frac{\lambda r n!}{(n-r)!} p^{r-1}, \quad q = 1-p, 1 \le r \le n \tag{2.7}$$

if and only if X has Poisson-binomial distribution with *pmf*

$$P(X = x) = e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^{m}}{m!} \binom{nm}{x} p^{x} q^{nm-x}, \ x = 0,1,\cdots,nm. \tag{2.8}$$

Proof: Suppose X has Poisson-binomial distribution with *pmf* (2.8), then by definition

$$G(1+t) = e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^{m}}{m!} \left[\sum_{x=0}^{nm} \binom{nm}{x} \left(p(1+t)\right)^{x} q^{nm-x} \right],$$

$$= e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^{m}}{m!} \left(q + p(1+t)\right)^{nm},$$

$$= e^{-\lambda} \sum_{m=0}^{\infty} \frac{1}{m!} \left[\lambda\left(1+pt\right)^{n}\right]^{m}, \ = e^{-\lambda} \exp\left[\lambda\left(1+pt\right)^{n}\right],$$

$$\ln G(1+t) = -\lambda + \lambda(1+pt)^{n},$$

$$= \lambda\left[npt + n(n-1)\frac{(pt)^{2}}{2!} + n(n-1)(n-2)\frac{(pt)^{3}}{3!} + \cdots + (pt)^{n}\right]$$

$$\ln G(1+t) = \sum_{r=1}^{n} \frac{\lambda n! p^{r}}{(n-r)!} \frac{t^{r}}{r!},$$

the coefficient of $t^{r}/r!$ is

$$\kappa_{[r]} = \lambda n!/(n-r)! p^{r}, 1 \le r \le n,$$

differentiating w.r.t 'p' we get (2.7).

Suppose (2.7) holds and after integration we have

$$\kappa_{[r]} = \frac{\lambda n!}{(n-r)!} p^{r} + C_{r}, \ 1 \le r \le n,$$

Since $\lim_{p\to 0} \ln G(z) = 0$, the limiting *fcgf* is

$$\lim_{p\to 0} \sum_{r=1}^{n} \left(\frac{\lambda n! p^{r}}{(n-r)!} + C_{r}\right)\frac{t^{r}}{r!} = 0,$$

which implies that $C_r = 0$, 1≤r≤n

Hence the *fcgf* is

$$\ln G(1+t) = \sum_{r=1}^{n} \frac{\lambda n! p^{r}}{(n-r)!} \frac{t^{r}}{r!},$$

$$= \lambda\left[npt + n(n-1)\frac{(pt)^{2}}{2!} + n(n-1)(n-2)\frac{(pt)^{3}}{3!} + \cdots + (pt)^{n}\right]$$

$$\ln G(1+t) = -\lambda + \lambda(1+pt)^{n}$$

$$G(1+t) = \exp\left(-\lambda + \lambda(1+pt)^{n}\right),$$

$$G(z) = e^{-\lambda} \exp\left(\lambda\left(1 + p(z-1)\right)^{n}\right),$$

as

$$P(X = x) = \left(\frac{1}{x!} \frac{d^{x}G(z)}{dz^{x}}\right)_{z=0}$$

and we get (2.8).

Corollary 2.2: Let

$$G(z) = \sum_{x=0}^{2m} z^{x} P(X = x)$$

be the probability generating function (*pgf*) of a distribution with support 0, 1, 2,…,2m, m$\in \mathbb{Z}^{+}$ and parameters λ, $\lambda>0$ and p, $0<p<1$ such that $\lim_{p\to 0}\ln G(z)=0$. Then

$$\frac{d\kappa_{[r]}}{dp}=\lambda r 2!/(2-r)!\,p^{r-1},\ r=1,2 \qquad (2.9)$$

if and only if X has Hermite distribution with *pmf*

$$P(X=x)=e^{-a}\sum_{m=0}^{\infty}\frac{a^m}{m!}\binom{2m}{x}p^x q^{2m-x},\ x=0,1,\cdots,2m \quad (2.10)$$

Proof: If we put n = 2 in Theorem 2.3, we get Corollary 2.2.

Theorem 2.4: Let

$$G(z)=\sum_{x=0}^{\infty}z^x P(X=x)$$

be the probability generating function (*pgf*) of a distribution with support 0,1,2,… and parameter θ, $\theta>0$, such that $\lim_{\theta\to 0}\ln G(z)=0$. Then

$$\frac{d\kappa_{[r]}}{d\theta}=1,\ \text{for } r=1 \qquad (2.11)$$

if and only if X has Poisson distribution with *pmf*

$$P(X=x)=\frac{e^{-\theta}\theta^x}{x!},\ x=0,1,.... \qquad (2.12)$$

Proof: Suppose that X follows Poisson distribution with *pmf* (2.12) and we have

$$G(1+t)=e^{-\theta}\sum_{x=0}^{\infty}\frac{(\theta(1+t))^x}{x!}=\exp(\theta t),$$

$$\ln G(1+t)=\theta t,$$

the coefficient of $t^r/r!$ is $\kappa_{[r]}=\theta$, for r = 1 and 0 otherwise, differentiating w.r.t 'θ' we get (2.11).
Suppose (2.11) holds and after integration we have

$$\kappa_{[r]}=\theta+C_r,\ r=1$$

where C_r is a constant of integration.
Since $\lim_{\theta\to 0}\ln G(z)=0$, the limiting *fcgf* is

$$\lim_{\theta\to 0}\sum_{r=1}^{\infty}(\theta+C_r)\frac{t^r}{r!}=0$$

which implies that $C_r = 0$, for r = 1
Hence the *fcgf* is

$$\ln G(1+t)=\theta t,$$

$$G(1+t)=\exp(\theta t),$$

$$G(z)=\exp(\theta(z-1))$$

as

$$P(X=x)=\left(\frac{1}{x!}\frac{d^xG(z)}{dz^x}\right)_{z=0}$$

and we get (2.12).

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