World Applied Sciences Journal 7 (11): 1446-1455, 2009
ISSN 1818-4952

Ideal Theory of BCH-Algebras

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Abstract: In this paper, we introduce the notions of atoms and some types of ideals in BCH-algebras and we stated and proved some theorems which determine the relationship between these ideals and other ideals of BCH-algebras and by some examples we show that these notions are different.

Key words: BCH-algebra . BCI/BCK-algebra . atom. ideal . implicative ideal . positive implicative ideal . fantastic ideals . p-ideal . positive ideal . normal ideal

2000 mathematics subject classification: 06F35 . 03G25

INTRODUCTION AND PRELIMINARIES

Definition 1.1: [2] By a BCH-algebra we shall mean an algebra (X,*,0) of type (2,0) satisfying the following axioms: for every x, y, z $\in$X,

(I1) $x * x = 0$,

(I2) $x * y = 0$ and $y * x = 0$ imply $x = y$,

(I3) $(x * y) * z = (x * z) * y$,

Proposition 1.2: [1, 2, 3] In a BCH-algebra X, the following holds for all x, y, z $\in$X,

(1) $x * 0 = x$,

(2) $(x * (x * y)) * y = 0$,

(3) $0 * (x * y) = (0 * x) * (0 * y)$,

(4) $0 * (0 * (0 * x)) = 0 * x$,

(5) $x \le y$ implies $0 * x = 0 * y$

[2] A BCH-algebra X is called proper if it is not a BCI-algebra. It is known that proper BCH-algebras exist.

In any BCH/BCI/BCK-algebra X we can define a partial order $\le$ by putting $x \le y$ if and only if $x*y=0$, [4, 5, 7].

Definition 1.3: [2] Let I be a nonempty subset of X. Then I is called an ideal of X if it satisfies:

(i) $0 \in I$
(ii) $x*y \in I$ and $y \in I$ imply $x \in I$

Definition 1.4: An ideal I is called a closed ideal of X if for every $x \in I$, we have $0*x \in I$.

Definition 1.5: [2] Let S be a subset of X. S is called a subalgebra of X if for every x, $y \in S$, we have $x*y \in S$.

ATOMS OF BCH-ALGEBRAS

From now on X is a BCH-algebra, unless otherwise is stated.

Definition 2.1: A BCH-algebra X is called medial if

$$(x * y) * (z * u) = (x * z) * (y * u)$$

for all x, y, z, $u \in X$.

Definition 2.2: A BCH-algebra X that satisfying in condition $0*x=0 \Rightarrow x=0$ is called a P-semisimple BCH-algebra.

Definition 2.3: A BCH-algebra X is called associative BCH-algebra if $(x*y)*z = x*(y*z)$, for all x, y, z, $u \in X$.

Definition 2.4: In a BCH-algebra X, define

$$X_+ = \{x \in X \mid x \ge 0\}$$

and

$$L_K(X) := \{a \in X_+ \setminus \{0\} \mid x \le a \Rightarrow x = a, \forall\ x \in X \setminus \{0\}\}$$

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$$L_P(X) := \{a \in X \mid x \le a \Rightarrow x = a \ \forall\ x \in X\}$$

$$L_I(X) := \{a \in X\backslash\{0\} \mid x \le a \Rightarrow x = a \ \forall\ x \in X\backslash\{0\}\}$$

$$L_P^*(X) := L_P(X)\backslash\{0\}$$

$$L_+(X) := L_K(X) \cup \{0\}$$

$$L\{X\} := L_I(X) \cup \{0\}.$$

Example 2.5: Let X= {0, a, b, c, d}. The following table shows the BCH-algebra structure on X

*	0	a	b	c	d
0	0	0	0	0	d
a	a	0	0	a	d
b	b	b	0	0	d
c	c	c	c	0	d
D	d	d	d	d	0

Then $L_K(X) = \{a\}$, $L_P(X) = \{0, d\}$ and $L_I(X) = \{a,d\}$.

Example 2.6: Let X = {0, 1, 2, 3}. The following table shows the BCH-algebra structure on X.

*	0	1	2	3
0	0	0	0	0
1	1	0	3	3
2	2	0	0	2
3	3	0	0	0

It is clear that $L_K(X) = \{3\}$, $L_P(X) = \{0\}$ and $L_I(X) = \{3\}$

Example 2.7: Let X={0,a,b,c,d,e,f,g,h,i,j,k,l,m,n}. The following table shows the BCH-algebra structure on X.

*	0	a	b	c	d	e	f	g	h	i	j	k	l	m	n
0	0	0	0	0	0	0	0	0	h	h	h	h	l	l	n
a	a	0	a	0	a	0	a	0	h	h	h	h	m	l	n
b	b	b	0	0	f	f	f	f	i	h	k	k	l	l	n
c	c	b	a	0	g	f	g	f	i	h	k	k	m	l	n
d	d	d	0	0	0	0	d	d	j	h	h	j	l	l	n
e	e	e	a	0	a	0	e	d	j	h	h	j	m	l	n
f	f	f	0	0	0	0	0	0	k	h	h	h	l	l	n
g	g	f	a	0	a	0	a	0	k	h	h	h	a	l	n
h	h	h	h	h	h	h	h	h	0	0	0	0	n	n	l
i	i	i	h	h	k	k	k	k	b	0	f	f	n	n	l
j	j	j	h	h	h	h	j	j	d	0	0	d	n	n	l
k	k	k	h	h	h	h	h	h	f	0	0	0	n	n	l
l	l	l	l	l	l	l	l	l	n	n	n	n	0	0	h
m	m	l	m	l	m	l	m	l	n	n	n	n	a	0	h
n	n	n	n	n	n	n	n	n	l	l	h	l	h	h	0

Then $L_K(X) = \{a,f\}$, $L_P(X) = \{0,h,l,n\}$

and $L_I(X) = \{a,f,h,l,n\}$.

Proposition 2.8: In X, the following properties hold:

(i) $L_P(X) = Med(X)$, where $Med(X) = \{x \in X \mid 0*(0*x) = x\}$ is the medial part of X,
(ii) $L_K(X) \cap L_I(X) = \phi$,
(iii) $L(X) = L_K(X) \cup L_P(X)$,
(iv) X is a P-semisimple BCH-algebra if and only if $L_P(X) = X$.

Proof: (i) Let $a \in L_P(X)$. Then $0*(0*a) = a$ follows from $(0*(0*a))*a = 0$. Hence $a \in Med(X)$.

Conversely, let $a \in Med(X)$ and $x \in X$ be such that $x*a=0$. Then

$$\begin{aligned} a*x &= (0*(0*a))*x = (0*x)*(0*a) \\ &= ((x*a)*x)*(0*a) = ((x*x)*a)*(0*a) \\ &= (0*a)*(0*a) = 0. \end{aligned}$$

Hence $a = x$ and $a \in L_P(x)$. Therefore $L_P(x) = Med(X)$.

(ii) If $a \in L_K(x) \cap L_P(x)$, then $a \in X_+ \cap L_P(x) = \{0\}$ and so $a = 0$, which is a contradiction. Hence $L_K(x) \cap L_P(x) = \phi$.

(iii) Straightforward.

(iv) Let X be P-semisimple. Since X = Med(X), it follows from (i) that $X = L_P(x)$.
Conversely, if $X = L_P(x)$ then Med(X) = X.

Remark 2.9: If X is a BCK-algebra, then $L(X) = L_+(x)$. But the converse is not true [6]. In Example 2.6 we have $X_+=X$ and $L(X) = \{0, 3\} = L_+(X)$. On the other hand we have $(0*2)*(0*1) = 0 \neq 3 = 1*2$. Hence X is not a BCK-algebra.

Definition 2.10: The elements of $L_K(X)$ (resp. $L_P^*(X)$, $L_I(X)$ are called a K-atom (resp. P-atom, I-atom) of X. For any $a \in X$. Let

$$V(a) = \{x \in X \mid a \le x\}$$

If $a \in L_K(X)$ (resp. $L_P(X)$, $L_I(X)$), we say that V (a) is K-branch (resp. P-branch, I-branch) of X with respect to a.

Note that {P-atoms} $\cup$ {K-atoms} = {I-atoms}. Obviously, $V(a) \subseteq V(0) = X_+$, for all $a \in L_K X$ and X $= \cup_{a \in L_P(X)} V(a)$. But $X^* \neq \cup_{a \in L_I(X)} V(a)$, where $X^* = X \backslash \{0\}$ as shown in the following example.

Example 2.11: Let X= {0, 1, 2, 3, 4}. The following table shows the BCH-algebra structure on X.

*	0	1	2	3	4
0	0	0	0	0	4
1	1	0	0	1	4
2	2	2	0	0	4
3	3	3	3	0	4
4	4	4	4	4	0

It is routine to check that $L_I(X) = \{1, 4\}$ and $L(X) = \{0, 1, 4\}$. It is clear that $\cup_{a \in L_I(X)} V(a) = \{1,2,4\} \neq X^*$.

Note: In the above example, we see that there exists $x \in X$ which is not contained in any I-branch of X.

Definition 2.12: If $X^* = \cup_{a \in L_I(X)} V(a)$, we call X the BCH-algebra generated by I-atoms.

Example 2.13: Let X = {0, 1, 2, 3, 4, 5}. The following table shows the BCH-algebra structure on X.

*	0	1	2	3	4	5
0	0	0	0	0	4	4
1	1	0	0	1	4	4
2	2	2	0	2	5	4
3	3	3	3	0	4	4
4	4	4	4	4	0	0
5	5	5	4	5	2	0

It is routine to check that $L_I(X) = \{1, 3, 4\}$ and V (1) = {1, 2}, V (3) = {3}, V (4) = {4, 5}. Thus

$$X^* = V(1) \cup V(3) \cup V(4)$$

and so X is a BCH-algebra generated by I-atoms {1,3,4}.

Lemma 2.14: $a \in L_K(X)$ if and only if $a \in L_I(X) \cap X_+$.

Proof: By definition we have $L_K(X) \subseteq X_+$ and $L_I(X)$. The converse is clear.

Lemma 2.15: If $a \in L_I(X)$ satisfies the condition $x*(x*a) \in X_+ \backslash \{0\}$ for some $x \in X$, then $a \in X_+$

From above lemma we have the following theorem.

Theorem 2.16: Let a be an I-atom of X which satisfies in the following condition $x*(x*a) \in X_+ \backslash \{0\}$ for some $x \in X$. Then a is a K-atom of X.

Proof: We have from Lemma 2.15, $a \in X_+$ also $a \in L_I(X)$ hence $a \in L_I(X) \cap X_+$. By Lemma 2.14, we get that $a \in L_K(X)$ so a is a K-atom.

We have a characterization of P-atom by I-atom.

Theorem 2.17: Let $a \in X$. Then a is a P-atom of X if and only if $a \in L_I(X)$ and $x*(x*a) \neq 0$, for all $x \in X$.

Proof: Let a be a P-atom and

$$a \in L_P^*(X) \subseteq L(X) = L_I(X) \cup \{0\}$$

Then $a \in L_I(X)$, since $(x*(x*a))*a = (x*a)*(x*a) = 0$ it follow that $x*(x*a) = a \neq 0$.

Conversely, let $a \in L_I(X)$ and $x*(x*a) \neq 0$, for all $x \in X$. We show that $a \notin L_K(X)$. If $a \in L_K(X)$, then $0*a=0$. So $0*(0*a) = 0$. Which is a contradiction.

The following theorem is a characterization of an I-atom in a BCH-algebra.

Theorem 2.18: Let $a \in X^*$ and $\overline{X(a)} := \{x \in X \mid x*(x*a) \neq 0\}$. Then the following conditions are equivalent:

(i) A is an I-atom of X,
(ii) $a = x*(x*a)$ for all $x \in \overline{X(a)}$,
(iii) $(x*y)*(x*a) = a*y$, for all $y \in X, x \in \overline{X(a)}$.

Proof: (i) $\Rightarrow$(ii) By $(x*(x*a))*a = 0$ and (i) we have $a = x*(x*a)$.

(ii) $\Rightarrow$(iii) By hypothesis we have $(x*y)*(x*a) = x*(x*a)*y = a*y$.

(iii)$\Rightarrow$(ii) Let $x \in \overline{X(a)}$ and y = 0. Then $x*(x*a) = (x*0)*(x*a) = a*0 = a$.

(ii)$\Rightarrow$(i) Let $b(\neq 0) \in X$. Since $w*a = 0$, then $w*(w*a) = w*0 = w \neq 0$ and so $w \in \overline{X(a)}$. It follow from (ii) that $a = w*(w*a) = w$. Therefore a is an I-atom of X.

Remark 2.19: [6] If X is a BCI-algebra, then the following conditions are equivalent:

(i) a is an I-atom of X,
(ii) $a = x*(x*a)$, for all $x\in X$
(iii) $(x*y)*(x*a) = a*y$ for all $y\in X, x\in \overline{X(a)}$.
(iv) $a*(x*z)\leq z*(x*a)$, for all $z\in X, x\in \overline{X(a)}$,
(v) $(a*y)*(x*z) \leq (z*y)*(x*a)$, for all $y,z\in X$, $x\in \overline{X(a)}$,
(vi) $x*(x*(a*y)) = a*y$, for all $y\in X, x\in \overline{X(a)}$.

We can see that these relations need not be true in BCH-algebras. Since in Example 2.7, if $x\in X^*$, then $\overline{X(a)} = \{a, c, g, m\}$ also a is an I-atom then (i) holds. If $x = g$ then

$$a*(g*l) = 0 \not\leq l = l*(g*a).$$

So (iv) does not hold. If $y = 0$, $x = g$ and $z = l$, then we have

$$(a*0)*(g*l) = a*(g*l),\ (l*0)*(g*a) = l*(g*a).$$

Then

$$a*(g*l) \neq l*(g*a).$$

Furthermore if $x = g \in \overline{X(a)}$ and $y = m$ then

$$g*(g*(a*m)) = g*(g*l) = g*a = f \neq l = a*m.$$

So that (v) is not true.

Theorem 2.20: If X is an associative BCH-algebra, then the conditions (i),(ii),(iii),(iv),(v),(vi) of remark 2.19 are hold.

Proof: Let X be an associative BCH-algebra. For all x, y, $z\in X$, we have

$$\begin{aligned}((x*y)*(x*z))*(z*y) &= ((x*(x*z))*y)*(z*y)\\ &= (((x*x)*z)*y)*(z*y) = ((0*z)*y)*(z*y)\\ &= (0*(z*y))*(z*y) = 0*(z*y)*(z*y)\\ &= 0*0 = 0.\end{aligned}$$

Then X is a BCI-algebra.

Corollary 2.21: For any nonzero element a of X, the following conditions are equivalent:

(i) a is a P-atom of X,
(ii) $a = x*(x*a)$ for all $x\in X$,
(iii) $(x*y)*(x*a) = a*y$, for all $x, y\in X$.

Proof: Let $a(\neq 0)\in X$. If a is a P-atom of X, then $(x*(x*a))*a = 0$ also a is an I-atom of X and $x*(x*a) = x\neq 0$, for all $x\in X$. Thus conditions (ii), (iii) follows from Theorem 2.18.

Conversely, assume that conditions (ii) and (iii) holds. Then we know that $x*(x*a) \neq 0$, for all $x\in X$. It follows from Theorem 2.18, that a is an I-atom of X. Hence by Theorem 2.17, a is a P-atom of X. This completes the proof.

Corollary 2.22: Let $a(\neq 0)\in X$ and

$$X(a)_+ := \{x\in X \mid x*(x*a)\geq 0,\ x*(x*a)\neq 0\} \neq \phi.$$

Then the following conditions are equivalent:

(i) a is a K-atom of X,
(ii) $a = x*(x*a)$, for all $x\in X(a)_+$,
(iii) $(x*y)*(x*a) = a*y$, for all $y\in X$, $x\in X(a)_+$.

Proof: Assume that a is a K-atom of X, then $a\in L_I(X)$ since $L_I(X) = L_I^*(X)\cup L_K(X)$. Note that $X(a)_+\subseteq \overline{X(a)}$, so from Theorem 2.18, we get that any one of (ii)-(iii) holds.

Conversely, if any one of the condition (ii)-(iii) holds, then $0\leq x*(x*a) \leq a$ for any $x\in X(a)_+$, i.e. $a\in X_+$. Let $y\in\overline{X(a)}$. Then $y*(y*a)\neq 0$, since $y*(y*a) \leq a$ we have $y*(y*a) \geq 0$. This show that $y\in X(a)_+$ and so $\overline{X(a)} = X(a)_+$. From Theorem 2.18 we get that $a\in L_I(X)$, so that $a\in L_I(X)\cap X_+ = L_K(X)$. Then a is a K-atom.

Theorem 2.23: Any finite BCH-algebra is generated by I-atoms.

Proof: Let X be a finite BCH-algebra and $x\in X^*$. Let

$$(x] := \{a\in X^* \mid a\leq x\}.$$

Then clearly $x\in (x]$ and so $(x]\neq\phi$. Hence we can take a minimal element of (x], say a_0. We claim that. $a_0\in L_I(X)$. For any $z\in X^*$, assume that $z*a_0 = 0$. Then $z\leq a_0\leq x$ and so $z\in (x]$. Since a_0 is a minimal element of (x] it follows that $z = a_0$. Hence $a_0\in L_I(X)$ and $x\in V(a_0)$. Therefore $X^* = \cup_{b\in L_I(X)} V(b)$.

Theorem 2.24: $L_P(X)$ and $L_+(X)$ are subalgebras of X.

Proof: Let $a,b\in L_P(X)$. We have $L_P(X) = Med(X)$, so $0*(0*a) = a$ and $0*(0*b) = b$, then

$$0*(0*(a*b)) = 0*((0*a)*(0*b))$$
$$= 0*(0*a))*(0*(0*b)) = a*b$$

It follows that $a*b \in Med(X) = L_P(X)$.

Note: The following example shows that L(X) may not be a subalgebra of X.

Example 2.25: Let X be BCH-algebra in Example 2.7. Then L(X) = {0, a, f, h, l, n} is not subalgebra of X, since a, l ∈ L(X), but a*l = m∉L(X)

Definition 2.26: An ideal I of X satisfies the following condition x∈ I and a∈X\I imply x*a∈I, is called a *-ideal of X.

Note: Every *-ideal is an ideal.

Lemma 2.27: Let I be an ideal of X. Then I is a closed *-ideal of X if and only if $L_P(X) \subseteq I$.

Theorem 2.28: In X, the following conditions are equivalent:

(i) Every nonzero element of X is a K-atom of X, i.e., $X = L_K(X) \cup \{0\}$,
(ii) x*y=x, for all x, y ∈ X with x≠y,
(iii) x*(x*y) = 0, for all x, y ∈ X with x≠y,
(iv) every subalgebra of X is a *-ideal of X.

Proof: (i)⇒(ii) Assume that (i) holds and let x, y ∈ X be such that x≠y. Then x*y≤x, since $y \in X = L_K(X) \cup \{0\}$. If x = 0, then obviously x*y=0 = x. Assume that x≠0. Then $x \in L_K(X)$. Note that x*y≠0, because if y = 0, then x*y=x≠0 and if y≠0 and x, $y \in L_K(X)$ we have $L_K(X)$ is a subset of X, then $0 \neq x*y \in L_K(X)$. Therefore x*y=x.
(ii) ⇒(iii) It is clear.
(iii)⇒(i) Assume that (iii) holds and x∈ X. If x = 0, then we are done. Suppose x≠0. Then by (iii), we have 0*(0*x) =0 therefore 0*x =0*(0*(0*x) = 0*0 = 0, then $X=X_+$ and so $L_I(X) = L_K(X)$.

Finally if $X \neq L_I(X) \cup \{0\}$ then there exists $z(\neq 0) \in X \setminus L_I(X)$ such that a*z=0, for some a(≠0), z∈X. It follows that a*(a*z) = a*0= a≠0, which is a contradiction. Therefore $X = L_I(X) \cup \{0\} = L_K(X) \cup \{0\}$.
(ii)⇒(iv). Let S be a subalgebra of X and x*y, y∈S. If x = y, then clearly x ∈ S. If x≠y, then x=x*y∈S. Hence S is an ideal of X.
(iv)⇒(i). Note that $L_+(X)$ is a subalgebra of X. It follows from (iv) that $L_+(X)$ is a *-ideal of X. Clearly $L_+(X)$ is closed. Hence $L_P(X) \subseteq L_+(X)$ and so $L_P(X)=\{0\}$ since $L_P(X) \cap L_+(X)=\{0\}$. This shows that $L(X) = L_+(X)$. Now let a be a nonzero element of X and 0≠z∈X be such that z*a=0. Note that S: = {0, a} is a subalgebra of X and hence S is a *-ideal of X by (iv). Since S is an ideal of X, it follows from z*a=0 that z∈ S and so z = a. This means that $a \in L_I(X) \subseteq L(X) = L_+(X)$. Since a≠0, it follows that $a \in L_K(X)$, i.e., a is a K-atom of X.

In the following we study branches of X. Note that $V(a) \cap V(b) \neq \phi$, for some a, b ∈ $L_I(X)$ with a≠b as shown in the following example.

Example 2.29: Let X = {0, 1, 2, 3, 4}. The following table shows the BCH-algebra structure on X.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

Then $L_I(X)$ = {1, 2, 4} and V (1) = {1, 3}, V (2) = {2, 3}. Therefore V (1) ∩V (2) = {3}.

Now, we want to define a proper I-branch of X and the proper I-branch BCH-algebra.

Definition 2.30: Let $a \in L_K(X)$. Then a branch V (a) is called a proper I-branch if for all $b \in L_I(X)$, $V(a) \cap V(b) \neq \phi$ whenever a≠b. If every I-branch of X is a proper I-branch of X, we say that X is a proper I-branch BCH-algebra.

Lemma 2.31: If $a \in L^*_P(X)$., then 0*(0*x) =a, for all x∈V(a).

Proof: Let $a \in L^*_I(X)$ and 0*(0*x) $=a_x$, for all x ∈V (a). Then $0*(0*a_x) = 0*(0*(0*(0*x))) = a_x$ and so $a_x \in Med(X) = L_P(X)$. Then

$$a*a_x = (0*(0*a))*(0*(0*x)) = 0*(0*(a*x)) = 0$$

Since $a_x \in L_P(X)$, it follows that $a = a_x = 0*(0*x))$.

Theorem 2.32: If a BCH-algebra X satisfies the following conditions:

(i) c*a=c for all $a \in L_K(X)$ and $c \in V(a) \setminus \{a\}$,
(ii) every subalgebra S of X with |S|≥3 is an ideal of X, then X is a proper I-branch BCH-algebra.

Proof: Let $a\in L_I(X)$. Then either $a\in L_K(X)$ or $a\in L_P^*(X)$. Consider the case $a\in L_P^*(X)$. We claim that $V(a)\cap V(b)=\phi$, for all $b\in L_I(X)$ with $a\neq b$. In fact, if $V(a)\cap V(b)\neq\phi$, for some $b\in L_I(X)$ with $a\neq b$, then there exits $c\in V(a)\cap V(b)$. It follows from above lemma that $0*(0*c)=a$. In this case, b must be in $L_P^*(X)$ because if $b\in L_K(X)$, then $0\leq b\leq c$ and so $c\in X_+$, which implies that $a=0*(0*c)=0$, which is a contradiction. By above lemma, we have $a=0*(0*c)=b$ which is a contradiction. Therefore $V(a)\cap V(b)=\phi$, for all $b\in L_I(X)$ with $a\neq b$.

Now, consider the remaining case, if $a\in L_K(X)$. Assume that $V(a)\cap V(b)\neq\phi$ for some $b\in L_I(X)$ with $a\neq b$, then there exists $c\in V(a)\cap V(b)$. If $c=a$, then $b\leq a$ implies $b=a$ since $a\in L_K(X)$. If $c\neq a$, then $c*a=c$ by (i). Hence $S=\{0, a, c\}$ is a subalgebra of X. It follows from (ii) that S is an ideal of X. Hence $b*c=0\in S$ and $c\in S$ imply $b\in S$. But $b\in L_I(X)$ implies $b\neq 0$ and so $b=c$ or $b=a$. Since c not in $L_I(X)$, it follows that $b=a$, which is a contradiction. This prove that every I-branch of X is proper so that X is a proper I-branch BCH-algebra.

Theorem 2.33: Let X be a proper I-branch BCH-algebra and $a,b\in L_I(X)$. Then $a*y=a$, for all $a\in L_P(X)$ and $y\in X_+$.

Proof: Let $a\in L_P(X)$ and $y\in X_+$. We have $a*y\leq a$, since $a\in L_P(X)$, then $a*y=a$.

SOME TYPES OF IDEALS IN BCH-ALGEBRA

Definition 3.1: A nonempty subset I of X is called a P-ideal of X if

(i) $0\in I$,
(ii) $(x*z)*(y*z)\in I$ and $y\in I$ imply $x\in I$, for all $x, y, z\in X$.

Proposition 3.2: Any P-ideal of X is an ideal of X.

Proof: Let I be a P-ideal, $x*y\in I$ and $y\in I$. Then $x*y=(x*0)*(y*0)\in I$ and $y\in I$ imply that $x\in I$.

The following example shows that the converse of above proposition is not correct in general.

Example 3.3: Let $X=\{0, 1, 2, 3, 4\}$. The following table shows BCH-algebra structure on X.

*	0	1	2	3	4
0	0	0	0	0	4
1	1	0	0	1	4
2	2	2	0	0	4
3	3	3	3	0	4
4	4	4	4	4	0

Then $I=\{0, 1\}$ is an ideal of X, but it is not a P-ideal, since $(2*2)*(1*2)=0\in I$ and $1\in I$, but $2\notin I$.

Lemma 3.4: If I is a P-ideal of X, then $X_+\subseteq I$

Proof: Let I be a P-ideal and $a\in X_+$. Then $0*a=0\in I$ and $(a*a)*(0*a)=0\in I$ since I is a P-ideal, therefore $a\in I$.

Remark 3.5: In BCI-algebra converse of above lemma is true but in BCH-algebra is not true. In Example 2.7 it is routine to show that $I=\{0, a, b, c, d, e, f, g\}$ is an ideal and $X_+\subseteq I$. Which is not a P-ideal because $(m*l)*(g*l)=a*a=0\in I$ and $g\in I$, but $m\notin I$.

Theorem 3.6: Every nonzero element of X is a P-atom if and only if every subalgebra of X is a P-ideal of X.

Proof: Assume that every nonzero element of X is a P-atom and S is a subalgebra of X. Since $X=L_P(X)$, therefore X is P-semisimple and hence is medial. It follows from Definition 2.1 that $(x*y)*(0*y)=(x*0)*(y*y)=x$ and

$$(x*z)*(y*z)=(x*y)*(z*z)=(x*y)*0=x*y$$

Let $(x*z)*(y*z)\in S$ and $y\in S$, for all $x, y, z\in X$. Then

$$x=(x*y)*(0*y)=((x*z)*(y*z))*(0*y)\in S$$

Therefore S is a P-ideal of X.

Conversely, suppose that every subalgebra of X is a P-ideal of X. Since $L_P(X)$ is a subalgebra of X and so is a P-ideal of X. Then we get that $X_+\subseteq L_P(X)$. Note that $X_+\cap L_P(X)=\{0\}$, so that $X_+=\{0\}$ and $X=L_P(X)$. This implies that every nonzero element of X is a P-atom of X.

Definition 3.7: A nonempty subset I of X is called an implicative ideal of X if

(i) $0\in I$,
(ii) $(x*(y*x))*z\in I$ and $z\in I$ imply $x\in I$, for all $x, y, z\in X$.

Proposition 3.8: Any implicative ideal of X is an ideal of X.

Proof: Let I be an implicative ideal, $x*z\in I$ and $z\in I$. Then $x*z=(x*0)*z=(x*(x*x))*z\in I$ and $z\in I$ imply $x\in I$.

Every ideal need not be an implicative ideal as shown in the following example.

Example 3.9: In Example 2.7, we can check that I = {0, a, b, c, d, e, f, g} is an ideal of X but $(h*(0*h))*a = (h*h)*a = 0 \in I$ and $a \in I$, but $h \notin I$. Hence I is not an implicative ideal.

By the following examples we show that notions of implicative ideal and P-ideal are independent.

Example 3.10: Let X = {0, a, b}. The following table shows the BCH-algebra structure on X.

*	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Now, we can see that I = {0, a} is an implicative ideal but is not P-ideal, because $(b*b)*(0*b)=0 \in I$ and $0 \in I$, but $b \notin I$.

Example 3.11: Let X = {0, a, b, c}. The following table shows the BCH-algebra structure on X.

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then I = {0, a} is a P-ideal, also $(b*(0*b))*a = (b*b)*a = a \in I$ and $a \in I$, but $b \notin I$. So I is not an implicative ideal.

Theorem 3.12: Let I be an ideal of X. Then I is an implicative ideal if and only if $x*(y*x) \in I$ imply that $x \in I$.

Proof: Assume that I is an implicative ideal and $x*(y*x) \in I$. Consider $(x*(y*x))*(x*(y*x)) = 0 \in I$, by hypothesis we get that $x \in I$.

Conversely, let I be an ideal. Now, let $(x*(y*x))*z \in I$ and $z \in I$ and so $x*(y*x) \in I$, By hypothesis $x \in I$, therefore I is an implicative ideal.

Theorem 3.13: Any nonzero element of X is a K-atom if and only if every subset of X is an implicative ideal of X.

Proof: Assume that every nonzero element of X is a K-atom, hence $X = L_K(X)$ and I are subalgebras of X. We have $(x*(y*x))*x = 0*(y*x) = (0*y)*(0*x) = 0$ for all $x, y \in X$ Since x is a K-atom therefore $x = x*(y*x)$. Now, consider $(x*(y*x))*z \in I$ and $z \in I$, then $(x*(y*x))*z = x*z \in I$, for all $x, y, z \in X$. Also we have $(x*z)*x = 0*z = 0$. Since x is a K-atom therefore $x*z = x$, hence $x \in I$. So S is an implicative ideal of X.

Conversely, suppose that every subalgebra of X is an implicative ideal of X. We first show that $0*x = 0$ for all $x \in X$. Since A= {0} is nonempty subalgebra of X, therefore A is an implicative ideal. We deduce that

$$\begin{aligned}
&(0*x)*(((0*x)*x)*(0*x)) \\
&=(0*(((0*x)*x)*(0*x)))*x \\
&=((0*((0*x)*x))*(0*(0*x)))*x \\
&=(((0*(0*x))*(0*x))*(0*(0*x)))*x \\
&=(0*(0*x))*x \\
&=(0*x)*(0*x)=0
\end{aligned}$$

Hence $((0*x)*(((0*x)*x)*(0*x)))*0=0 \in A$ and $0 \in A$ imply $0*x \in A$. Therefore $0*x=0$. Further for $x \in X \backslash \{0\}$, we get that I = {0, x} is subalgebra of X, then is an implicative ideal. Now let $y*x=0$ for some $y \in X \backslash \{0\}$, then $y*x = (y*(y*y))*x = 0 \in I$ and $x \in I$ imply that $y \in I$. Hence y = x. Therefore x is a K-atom.

Definition 3.14: A nonempty subset I of X is called a positive implicative ideal of X if

(i) $0 \in I$,
(ii) $(x*y)*z \in I$ and $y*z \in I$ imply $x*z \in I$ for all $x, y, z \in X$.

Proposition 3.15: Every positive implicative ideal is an ideal.

Proof: Let $x*y \in I$ and $y \in I$. Then $(x*y)*0 = x*y \in I$ and $y*0 = y \in I$, by hypothesis we get that $x = x*0 \in I$. Hence I is an ideal.

Example 3.16: Let X = {0, 1, 2, 3}. The following table shows the BCH-algebra structure on X.

*	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	3
3	3	3	3	0

Then I = {0, 3} is an ideal of X, but is not a positive implicative ideal, since $(2*3)*1 = 3 \in I$ and $3*1 = 3 \in I$ but $2*1 = 2$ is not in I.

Remark 3.17: The next examples shows that the notions positive implicative ideal, implicative ideal and P-ideal are independent.

(1) Let X = {0, a, b, e}. The following table shows the BCH-algebra structure on X.

*	0	a	b	e
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
e	e	e	e	0

We can check that I = {0, a} is a positive implicative ideal which is not an implicative ideal, since (b*(e*b))*a = (b*e)*a = 0∈I and a ∈I, but $b \notin I$.
Also I is not a P-ideal, since

$$(b*b)*(a*b) = 0*a = 0 \in I$$

and a ∈I, but $b \notin I$

(2) In example 3.11, consider I = {0, a}. Then I is a P-ideal, but is not a positive implicate ideal, since

$$(a*b)*b = c*b = a \in I$$

and b*b = 0∈I, but a*b = c is not in I.

Remark 3.18: We showed if every subset I of X is an implicative ideal, then every nonzero element a of X is K-atoms. Also if every subset I of X is P-ideal then a is P-atoms.

In remark 3.16 we showed that a positive implicative ideal is not necessary implicative ideal or P-ideal. So there is no relationship between K-atoms and P-atoms and positive implicative ideal.

Definition 3.19: An ideal I of X is called a normal ideal if x*(x*y)∈I implies y*(y*x)∈I, for all x, y∈X.

Example 3.20: Let X = {0, 1, 2, 3}. The following table shows the BCH-algebra structure on X.

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	3	0	3
3	3	0	0	0

Then I = {0, 3} is an ideal of X. But I is not a normal ideal, since 2*(2*1) ∈I, but $1*(1*2) \notin I$.

In the following example, we show that every normal ideal need not be a P-ideal, implicative ideal or positive implicative ideal.

Example 3.21: Let X = {0, a, b, c}. The following table shows the BCH-algebra structure on X.

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	b	0

Then I = {0, a} is a normal ideal. But I is not a P-ideal, since (b*b)*(0*b) = 0∈I and 0∈ I, but $b \notin I$.

Also I is not an implicative ideal, since (b*(c*b))*0 = (b*b)*0 = 0∈I and 0 ∈ I, but b is not in I.

I is not a positive implicative ideal, since (c*b)*b = b*b = 0∈I and b*b = 0 ∈I, but $c*b = b \notin I$.

Proposition 3.22: Let I be a normal ideal of X. Then I is a closed ideal.

Proof: Consider (0*x)*((0*x)*0) = (0*x)*(0*x) = 0∈I. Since I is a normal ideal, we get that 0*x = (0*(0*(0*x))) ∈I.

Remark 3.23: The converse of above proposition is not true in general, since in Example 3.20, consider the ideal I = {0, 3}, which is a closed ideal but is not a normal ideal.

Definition 3.24: An element x ∈X is called positive if 0≤x and an ideal I of X is called a positive ideal of X if any x∈ I be positive.

In the following example, we show that every positive ideal need not be a P-ideal or positive implicative ideal.

Example 3.25: Let X = {0, 1, 2, 3}. The following table shows the BCH-algebra structure on X.

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	3	0	3
3	3	0	0	0

(1) We can check that I = {0, 3} is a positive ideal.
(2) I is not a P-ideal, since (1*1)*(0*1) = 0∈I and 0∈I, but 1 is not in I.
(3) Also I is not a positive implicative ideal, since (1*2)*3 = 0∈I and 2*3 = 3∈I, but $1*3=1\notin I$.

In the following example we show that a normal ideal need not be a positive ideal.

Example 3.26: Let X = {0, 1, 2, 3}. The following table shows the BCH-algebra structure on X.

*	0	1	2	3
0	0	0	2	2
1	1	0	2	2
2	2	2	0	0
3	3	2	1	0

Then I = {0, 1, 2} is a normal ideal but is not a positive ideal, since 0*2=2≠0.

Definition 3.27: A nonempty subset I of X is called a fantastic ideal if

(i) 0∈I,
(ii) (x*y)*z∈I and z ∈I imply (x*(y*(y*x)))∈I, for all x, y, z ∈X.

Lemma 3.28: Let I be an ideal of X. Then I is a fantastic ideal if and only if x*y∈I implies (x*(y*(y*x)))∈I.

Proof: Let I be a fantastic ideal and (x*y) = (x*y)*0∈I. Since 0∈I then x*(y*(y*x))∈I.

Conversely, let (x*y)*z∈I and z∈I. Since I is an ideal, then x*y∈I. By hypothesis we get that (x*(y*(y*x))) ∈I.

Definition 3.29: An ideal I of X is called an obstinate ideal if x, y not in I imply x*y∈I or y*x∈I.

Remark 3.30: Every ideal is not an obstinate ideal because in Example 2.7, consider the ideal I = {0, a, b, c, d, e, f, g} of X. We can check that $h,l\notin I$. $h*l=n\notin I$ and $l*h=n\notin I$.

Proposition 3.31: Let x*(y*(y*x)) = x*y, for all $x,y\notin I$. Then {0} is a fantastic ideal of X.

Proof: Let x*y∈I = {0}. If x = 0 then x*(y*(y*x)) = 0*y*(y*0)) = 0∈I. therefore by Lemma 3.28 we conclude that I is a fantastic ideal. If y = 0 then x = x*0 = x*y = 0. Hence x*(y*(y*x)) = 0∈I, therefore by lemma 3.28 we conclude that I is a fantastic ideal. If x,y≠0, by hypothesis x*(y*(y*x)) = x*y∈I, then I is a fantastic ideal.

We show in the following example that the converse of above proposition is not true in general.

Example 3.32: Let X = {0, 1, 2, 3}. The following table shows the BCH-algebra structure on X.

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then {0} is a fantastic ideal, but 1*(2*(2*1)) = 0 ≠ 3 = 1*2.

In the following example, we show that every obstinate ideal need not be a P-ideal, implicative ideal or positive implicative ideal.

Example 3.33: Let X = {0, 1, 2, 3}. The following table shows the BCH-algebra structure on X.

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	3	0	3
3	3	0	0	0

Then I = {0, 3} is an obstinate ideal of X. We know that (1*2)* (0*2) = 0∈I and 0 ∈I, but 1∉I then I is not a P-ideal.

Also (2* (3*2))*3∈I and 3 ∈I, but 2∉I, then I is not an implicative ideal.

I is not positive implicative ideal, since (1*2)*3 = 0∈I and 2*3 = 3∈I, but $1*3=1\notin I$.

CONCLUSION

We introduced the notion of K-atoms, P-atom and I-atom and research relations between them. Also we introduced the notion of (P, implicative, positive implicative, normal, positive, obstinate and fantastic) ideals in BCH-algebras and gave characterizations of (P, implicative, positive implicative, normal, positive, obstinate and fantastic) ideals. We also studied the relations between P-ideals, implicative ideals, positive implicative ideal, positive ideal, obstinate ideal, normal ideals and fantastic ideals.

ACKNOWLEDGEMENT

The authors would like to express their sincere thanks to the referees for their valuable suggestions and comments.

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