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Linear Programming for the Location Problem with Minimum Absolute Error

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Abstract: Let n points be given in the plane. For each point a weight and a radius is given. The radius of each point is our expected ideal distance from the point to a new facility. In this paper we consider the location problem with minimum absolute errors. In this problem we want to find the location of the new facility such that the sum of weighted absolute errors between points and this new facility is minimized. We consider the case that the distances in the plane are measured by the block norms and show that it can be modeled by linear programming.

Key words: Location theory . distance constraints . minisum . block norm

INTRODUCTION

Facility location problems have several applications in telecommunications, industrial transportation and distribution, etc. Location problems in the most general form can be stated as follows. A set of customers spatially distributed in a geographical area originates demands for some kind of goods or services. Customers demand must be supplied by one or more facilities, which can operate in a cooperative or competitive framework, depending on the type of good or service being required. The decision process must establish where to locate the facilities in the territorial space taking into account users requirements and possible geographical restrictions. Each particular choice of facility site implies some set up cost for establishing the facility and some operational costs for serving the customers.

Location theory is branched to three major parts: continuous, discrete and network location problems. In the continuous location models the location of facilities in some d-dimensional space R^d is to be planned [1]. In the discrete case facilities may only be located at some existing points [2]. And in the network location models it may be useful to further distinguish between models in general networks as opposed to specialized networks, like tree and path [3, 4, 5].

In this paper we consider an special continuous location problem. Let n points, called facilities, be given in the plane. The single facility location problem asks to find location of a new facility such that the sum of the weighted distances from all facilities to the new one is minimized. Now consider the case that it is ideal that the distance of any point p_i to the new facility exactly be r_i. For example if $r_i = r$ for I = 1,…,n and all points are lie on a circle then the center of circle is the solution. However in the most cases the location of this ideal facility does not exist. So we try to minimize the sum of the weighted absolute errors. This problem introduced by Fathali *et al*. [6]. Application of this model can be in locating powerhouses, warehouses, dumping, emergency services, plants, silos, stadiums, etc.

Many authors consider the location problems with distance constraints in which the distances from the new facility and points should be less than or equal to a given numbers. Another related problem is covering problem which try to find minimum number of facilities such that the distance from any point p_i to the closest facility is less than or equal to r_i. For more details in these two cases we refer the reader to the book of Francis *et al*. [1].

In what follows, in sections 2 and 3 we define the problem and show the optimal solution is in a rectangle contains all existing points, respectively. In section 4 two linear programming models are written for the problem with block norms. Although l_1 and l_∞ norms are two block norms, however we write an simpler linear programming for the problem with these norms in sections 5 and 6, respectively. Section 7 contains some examples for each cases.

PROBLEM FORMULATION

Let n points $p_i = (a_i, b_i)$, i = 1,…,n in the plane be given. Every point p_i corresponds to a client and has a weight w_i, Let r_i be the given ideal distance from p_i to the new facility. The distance between any two points x and y in the plane is denoted by d (x, y). Let e (x, p_i) = $|d(x, p_i)-r_i|$ be the *absolute error* between the points x

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and p_i. In this paper we want to find location of a point x in the plan such that the sum of the weighted absolute errors over all points is minimized, i.e.:

$$\min F(x) = \sum_{i=1}^{n} w_i e(x, p_i) \qquad (P_1)$$

REGION OF THE SOLUTION

In the most of the continuous location problems the convex hull or rectangular hull of the existing points is contain the optimal solution. The convex hull and rectangular hull of a set of points is defined as the smallest convex set and rectangle (with sides parallel to the x and y axes) containing the set, respectively. In this section we show that the optimal solution of problem P_1 lies in a rectangle contains all points which can be larger than their convex hull and rectangular hull.

Definition 3.1: Let $p_1, p_2, \ldots, p_n$ be n points in the plane. We define the new points as

$$RH_1 = (a_{min}, b_{min}),\ RH_2 = (a_{min}, b_{max}),$$
$$RH_3 = (a_{max}, b_{min}),\ RH_4 = (a_{max}, b_{max})$$

whose coordinates are

$$a_{min} = \min\{a_i - r_i | i = 1, \ldots, n\}$$

$$a_{max} = \max\{a_i + r_i | i = 1, \ldots, n\}$$

$$b_{min} = \min\{b_i - r_i | i = 1, \ldots, n\}$$

$$b_{max} = \max\{b_i + r_i | i = 1, \ldots, n\}$$

Theorem 3.2: Let $p_1, p_2, \ldots, p_n$ be n given points in the plane. The rectangular hull of RH_1, RH_2, RH_3, RH_4, contains the optimal solution for the problem (P_1).

Proof: Let B be the rectangular hull of the points RH_1, RH_2, RH_3, RH_4 and $x = (x_1, x_2)$ be a point out of B. We show that x is not an optimal solution. We present the proof for the case $x_1 > a_{max}$, the proof for the other cases will be the same. Let $x' = (a_{max}, x_2)$. For each $p_i = (a_i, b_i)$, $i = 1, \ldots, n$, we have

$$d(x, p_i) - r_i > d(x', p_i) - r_i \geq 0$$

Therefor since $w_i \geq 0$

$$F(x) = \sum_{i=1}^{n} w_i |d(x, p_i) - r_i| > \sum_{i=1}^{n} w_i |d(x', p_i) - r_i| = F(x')$$

So x is not an optimal solution.

THE PROBLEM WITH BLOCK NORMS

The block norms are norms whose contours are polytopes. For example l_1 and l_∞ are two block norms. The block norms first time are used to solve the location problems by Ward and Wendell [7,8]. They showed a block norm can be characterized as follows:

$$||x||_B = \min\{\sum_{g=1}^{r} |\lambda_g| : x = \sum_{g=1}^{r} \lambda_g b_g\} \qquad (1)$$

where the points b_g and $-b_g$ with $g = 1, \ldots, r$ form the extreme points of the polytope corresponding to the unit contour. They also presented another characterization based on polar set for block norms. This characterization follows:

$$||x||_B = \max\{|x b_g^0| : g = 1, \ldots, r^0\} \qquad (2)$$

where b_g^0 and $-b_g^0$ with $g = 1, \ldots, r^0$ are extreme points of the polar set

$$B^0 = \{ v : b_g v \leq 1 \text{ for all } g = \pm 1, \pm 2, \ldots, \pm r\}$$

By using these characterization Ward and Wendell [7,8] show that the minimax and minisum single facility location problems can be written as a linear programming.

If a block norm B, is applied to measure the distances in the plane then we have $d(x, p_i) = \|x - p_i\|_B$ and the problem (P_1) as follows

$$\min F(x) = \sum_{i=1}^{n} w_i \left| ||x - p_i||_B - r_i \right| \qquad (3)$$

This problem can be written as the linear programming in two ways:

1 For $i = 1, \ldots, n$ let

$$||x - p_i||_B = \sum_{g=1}^{r} (\lambda_{gi}^{+} + \lambda_{gi}^{-})$$

$$x - p_i = \sum_{g=1}^{r} (\lambda_{gi}^{+} - \lambda_{gi}^{-}) b_g$$

and $(z_i^+ - z_i^-) = ||x - p_i||_B - r_i$ then we can write the problem 3 as follows

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n} w_i(z_i^+ + z_i^-) \\
\text{s.t.} \quad & \\
& \sum_{g=1}^{r}(\lambda_{gi}^+ + \lambda_{gi}^-) - z_i^+ + z_i^- = r_i \quad i=1,\dots,n \\
& x - p_i = \sum_{g=1}^{r}(\lambda_{gi}^+ - \lambda_{gi}^-)b_g \quad g=1,\dots,r \quad i=1,\dots,n \\
& z_i^+, z_i^- \geq 0 \quad i=1,\dots,n \\
& \lambda_{gi}^+, \lambda_{gi}^- \geq 0 \quad g=1,\dots,r \quad i=1,\dots,n
\end{aligned} \tag{4}$$

2. For i = 1,…,n let

$$y_i = \|x - p_i\|_B = \max\{|(x-p_i)b_g^0| : g=1,\dots,r^0\}$$

and $z_i^+ - z_i^- = y_i - r_i$ then we have

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n} w_i(z_i^+ + z_i^-) \\
\text{s.t.} \quad & \\
& z_i^+ - z_i^- = y_i - r_i \quad i=1,\dots,n \\
& (x-p_i)b_g^0 \leq y_i \quad g=1,\dots,r^0 \quad i=1,\dots,n \\
& -(x-p_i)b_g^0 \leq y_i \quad g=1,\dots,r^0 \quad i=1,\dots,n \\
& z_i^+, z_i^-, y_i \geq 0 \quad i=1,\dots,n
\end{aligned} \tag{5}$$

Note that the first model contains 2n+2rn variables and 2n constraints, whereas the second problem contains 3n+2 variables and $4r^0n+n$ constraints.

THE PROBLEM WITH RECTILINEAR NORM

In this section we consider the model (P_1) for the case that the norm in the plane is rectilinear. The model (P_1) with rectilinear norm as follows:

$$\min F(x) = \sum_{i=1}^{n} w_i \, \big| |x_1 - a_i| + |x_2 - b_i| - r_i \big|$$

In this case we can write the model as the following linear programming.

For $i = 1,\dots,n$ let $u_i = |x_1 - a_i|$, $v_i = |x_2 - b_i|$ and $z_i = |u_1 + v_i - r_i|$ then we add $u_i \geq |x_1 - a_i|$, $v_i \geq |x_2 - b_i|$ and $z_i \geq |u_i + v_i - r_i|$ to constraints which they can be written by two constraints as in following model. In this case because of objective is minimization, in the optimal solution equality occurs. Linearization of problem with rectilinear norm have not terms u_i and v_i in objective function, so the model does not guaranty optimal solution of (6) satisfies in constraints with equality. So we have the following problem

$$\begin{aligned}
\min f_1 = \sum_{i=1}^{n} w_i z_i + u_i + v_i & \\
\text{s.t.} & \\
\left.\begin{aligned}
u_i &\geq x_1 - a_i \\
u_i &\geq a_i - x_1 \\
v_i &\geq x_2 - b_i \\
v_i &\geq b_i - x_2 \\
z_i &\geq u_i + v_i - r_i \\
z_i &\geq r_i - u_i - v_i \\
z_i, &\ u_i, v_i \geq 0.
\end{aligned}\right\} & \ i=1,\cdots,n
\end{aligned} \tag{6}$$

Note that this model is equivalent and they have the same optimal solution, however their value of objective functions differ.

THE PROBLEM WITH l_∞ NORM

Now consider the model (P_1) for the case that the norm in the plane is l_∞. In this case the problem as follows:

$$\min F(x) = \sum_{i=1}^{n} w_i \, | \max(|x_1 - a_i|, |x_2 - b_i|) - r_i |$$

For writing the linear programming of this model let For i = 1,…,n

$$y_i = \max(|x_1 - a_i|, |x_2 - b_i|)$$

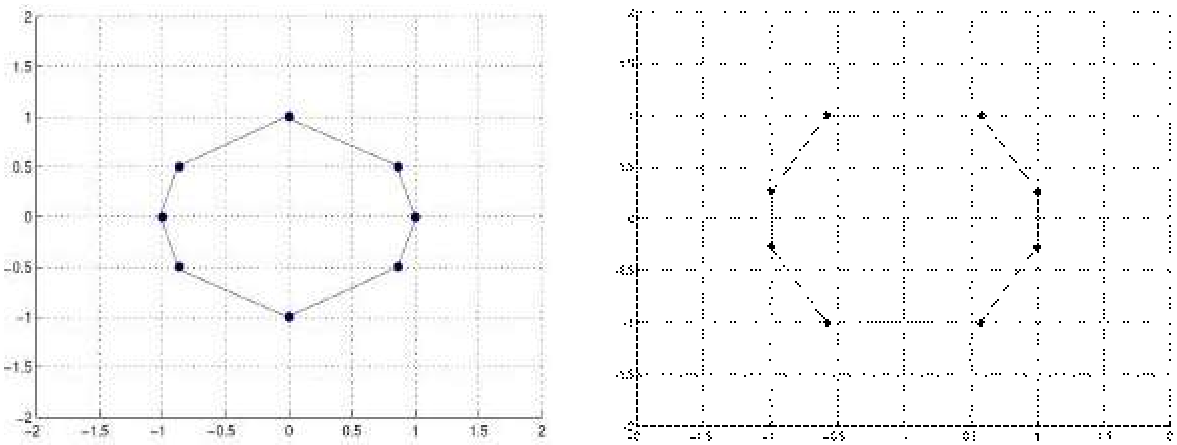
and $z_i = |y_i - r_i|$ and with the explanations of model with rectilinear norm, we have the following problem

$$\begin{aligned}
\min f_2 = \sum_{i=1}^{n} w_i z_i + y_i & \\
\text{s.t.} & \\
\left.\begin{aligned}
z_i &\geq y_i - r_i \\
z_i &\geq r_i - y_i \\
y_i &\geq x_1 - a_i \\
y_i &\geq a_i - x_1 \\
y_i &\geq x_2 - b_i \\
y_i &\geq b_i - x_2 \\
z_i, &\ y_i \geq 0.
\end{aligned}\right\} & \ i=1,\cdots,n
\end{aligned} \tag{7}$$

Because the objective function is minimization, exactly one of the $|x_1 - a_i|$, $|x_2 - b_i|$ will be selected in the optimal solution and also in $z_i \geq |y_i - r_i|$ equality occurs in optimal solution.

EXAMPLES

Example 7.1: Let the points $p_1 = (0,0)$, $p_2 = (0,1)$, $p_3 = (1,0)$ and $p_4 = (1,1)$ with weights $w_1 = w_2 = w_3 =$



(a) (b)

Fig. 1: The contour of block norms in the example, (a)Contour of block norm (b)Contour of polar set of block norm.

$w_4 = 1$ and radiuses $r_1 = r_2 = r_3 = r_4 = 1$ be given. Then the linear programming model of the problem in each case as follows.

1. Let the distances are measured by a block norm that the extreme points of its contour are $b_1 = (0,1)$, $b_2 = (\sqrt{3}/2, 1/2)$, $b_3 = (1,0)$, $b_4 = (\sqrt{3}/2, -1/2)$, $b_{-1} = (0,-1)$, $b_{-2} = (-\sqrt{3}/2, -1/2)$, $b_{-3} = (-1,0)$ and $b_{-4} = (-\sqrt{3}/2, 1/2)$ (Fig. 1a).

The extreme points of polar set of this block norm are $b_1^0 = (\sqrt{3}/3, 1)$, $b_2^0 = (1, 2-\sqrt{3})$, $b_3^0 = (1, -2+\sqrt{3})$, $b_4^0 = (\sqrt{3}/3, -1)$, $b_{-1}^0 = (-\sqrt{3}/3, -1)$, $b_{-2}^0 = (-1, -2+\sqrt{3})$, $b_{-3}^0 = (-1, 2-\sqrt{3})$ and $b_{-4}^0 = (-\sqrt{3}/3, 1)$ (Fig. 1b).
Now by using the model (4) we have

$$\min\ z_1^+ + z_1^- + z_2^+ + z_2^- + z_3^+ + z^- + z_4^+ + z_4^-$$

s.t

$$\sum_{g=1}^{4}(\lambda_{g1}^+ + \lambda_{g1}^-) - z_1^+ + z_1^- = 1 \quad , \quad x - p_1 = \sum_{g=1}^{4}(\lambda_{g1}^+ - \lambda_{g1}^-) b_g$$

$$x - p_2 = \sum_{g=1}^{4}(\lambda_{g2}^+ - \lambda_{g2}^-) b_g \quad , \quad x - p_3 = \sum_{g=1}^{4}(\lambda_{g3}^+ - \lambda_{g3}^-) b_g$$

$$x - p_4 = \sum_{g=1}^{4}(\lambda_{g4}^+ - \lambda_{g4}^-) b_g$$

$$z_i^+, z_i^- \geq 0 \quad i=1,2,3,4 \quad , \quad \lambda_{gi}^+, \lambda_{gi}^- \geq 0 . g=1,2,3,4$$

using this model we obtain:

$x^* = (0.5, 0.71133)$, $\lambda_{11}^+ = 0.4226497$, $\lambda_{21}^+ = 0.5773503$
$\lambda_{22}^+ = 0.4226497$, $\lambda_{42}^+ = 0.1547005$, $\lambda_{13}^+ = 0.4226497$
$\lambda_{44}^+ = 0.1547005$, $\lambda_{12}^- = 0.4226497$, $\lambda_{43}^- = 0.5773503$
$\lambda_{14}^- = 0.2113249$, $\lambda_{34}^- = 0.6339746$

and other variables are zero, so $F(x^*) = 0$.

2. If the distances are measured by rectilinear norm then by using model (6) we have the following linear programming

$$\min f_1 = z_1 + z_2 + z_3 + z_4 + u_1 + u_2 + u_3 + u_4 + v_1 + v_2 + v_3 + v_4$$

s.t.

$$\begin{array}{lll}
u_1 \geq & x_1 - 0, & u_1 \geq 0 - x_1 \\
u_2 \geq & x_1 - 0, & u_2 \geq 0 - x_1 \\
u_3 \geq & x_1 - 1, & u_3 \geq 1 - x_1 \\
u_4 \geq & x_1 - 1, & u_4 \geq 1 - x_1 \\
v_1 \geq & x_2 - 0, & v_1 \geq 0 - x_2 \\
v_2 \geq & x_2 - 1, & v_2 \geq 1 - x_2 \\
v_3 \geq & x_2 - 0, & v_3 \geq 0 - x_2 \\
v_4 \geq & x_2 - 1, & v_4 \geq 1 - x_2 \\
z_1 \geq & u_1 + v_1 - 1 & \\
z_1 \geq & 1 - u_1 - v_1 & \\
z_2 \geq & u_2 + v_2 - 1 & \\
z_2 \geq & 1 - u_2 - v_2 & \\
z_3 \geq & u_3 + v_3 - 1 & \\
z_3 \geq & 1 - u_3 - v_3 & \\
z_4 \geq & u_4 + v_4 - 1 & \\
z_4 \geq & 1 - u_4 - v_4 & \\
z_i, u_i, v_i \geq 0. & i=1,2,3,4 &
\end{array}$$

using this model we obtain:

$$x^* = (0.5, 0.5), z = (0,0,0,0),$$
$$u = v = (0.5, 0.5, 0.5, 0.5), f_1^* = 4$$

By putting this solution in the main problem we have $F(x^*) = 0$.

3. If the distances are measured by l_∞ norm then by using model (7) we have the following linear programming

$$\min f_2 = z_1 + z_2 + z_3 + z_4 + y_1 + y_2 + y_3 + y_4$$

s.t.

$$\begin{array}{ll}
z_1 \geq y_1 - 1, & z_1 \geq 1 - y_1 \\
z_2 \geq y_2 - 1, & z_2 \geq 1 - y_2 \\
z_3 \geq y_3 - 1, & z_3 \geq 1 - y_3 \\
z_4 \geq y_4 - 1, & z_4 \geq 1 - y_4 \\
y_1 \geq x_1 - 0, & y_1 \geq 0 - x_1 \\
y_2 \geq x_1 - 0, & y_2 \geq 0 - x_1 \\
y_3 \geq x_1 - 1, & y_3 \geq 1 - x_1 \\
y_4 \geq x_1 - 1, & y_4 \geq 1 - x_1 \\
y_1 \geq x_2 - 0, & y_1 \geq 0 - x_2 \\
y_2 \geq x_2 - 1, & y_2 \geq 1 - x_2 \\
y_3 \geq x_2 - 0, & y_3 \geq 0 - x_2 \\
y_4 \geq x_2 - 1, & y_4 \geq 1 - x_2 \\
z_i, y_i \geq 0. & i=1,2,3,4
\end{array}$$

using this model we obtain:

$$x^* = (0,0), z=(1,0,0,0), y=(0,1,1,1), f_2^* = 4$$

and the objective function of main problem is $F(x^*) = 1$.

SUMMARY AND CONCLUSION

We considered a new single facility location problem with best distance. In this problem we want to find location of a new facility such that the sum of weighted absolute errors be minimized. This type of single facility location problem has many applications in real life situations. In general, this problem is non convex and we showed the optimal solution of problem is in rectangle contains all points. We considered problem with block norm and write two linear programming models for it. l_∞ and l_1 are two block norms and we write a simpler linear programming for the problem with these norms.

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