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Bipolar-valued Fuzzy BCK/BCI-algebras

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Abstract: In this note, by using the concept of Bipolar-valued fuzzy set, the notion of bipolar-valued fuzzy BCK/BCI-algebra is introduced. Moreover, the notions of (strong) negative s-cut (strong) positive t-cut are introduced and the relationship between these notions and crisp subalgebras are studied.

Key words: Bipolar-valued fuzzy sets . Bipolar-valued fuzzy BCK/BCI-algebra . (strong) negative s-cut . (strong) positive t-cut

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INTRODUCTION

As it is well known, BCK/BCI-algebras are two classes of algebras of logic. They were introduced by Imai and Iseki [6-8]. BCI-algebras are generalizations of BCK-algebras Most of the algebras related to the t-norm based logic, such as MTL-algebras, BL-algebras [3, 4], hoop, MV-algebras and Boolean algebras *et al.* are extensions of BCK-algebras.

In 1965, Zadeh [13] introduced the notion of a fuzzy subset of a set; fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. Since then it has become a vigorous area of research in different domains, There have been a number of generalizations of this fundamental concept such as intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, soft sets etc [2].

Lee [10] introduced the notion of bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0,1] to [-1,1].

In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degree (0,1] indicates that elements somewhat satisfy the property and the membership degree [-1,0) indicates that elements somewhat satisfy the implicit counter-property. Bipolar-valued fuzzy sets and intuitionistic fuzzy sets look similar each other. However, they are different each other [10, 11].

Now, in this note we use the notion of Bipolar-valued fuzzy set to establish the notion of bipolar-valued fuzzy BCK/BCI-algebras; then we obtain some-related which have been mentioned in the abstract.

PRELIMINARIES

In this section, we present now some preliminaries on the theory of bipolar-valued fuzzy set. In his pioneer work [13], Zadeh proposed the theory of fuzzy sets. Since then it has been applied in wide varieties of fields like Computer Science, Management Science, Medical Sciences, Engineering problems etc. to list a few only.

Definition 2.1: [10] Let G be a nonempty set. A bipolar-valued fuzzy set B in G is an object having the form

$$B = \left\{\left(x, \mu^{+}(x), \nu^{-}(x)\right) \middle| \; x \in G\right\}$$

Where $\mu^{+}: G \to [0,1]$ and $\nu^{-}: G \to [-1,0]$ are mappings.

The positive membership degree $\mu^{+}(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set

$$B = \left\{\left(x, \mu^{+}(x), \nu^{-}(x)\right) \middle| \; x \in G\right\}$$

and the negative membership degree $\nu^{-}(x)$ denotes the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar-valued fuzzy set

$$B = \left\{\left(x, \mu^{+}(x), \nu^{-}(x)\right) \middle| \; x \in G\right\}$$

If $\mu^{+}(x) \neq 0$ and $\nu^{-}(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for

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$$B = \left\{ \left(x, \mu^{+}(x), \nu^{-}(x) \right) \middle| \; x \in G \right\}$$

If $\bar{\mu}(x)=0$ and $\bar{\nu}(x)\neq 0$, it is the situation that x does not satisfy the property of

$$B = \left\{ \left(x, \mu^{+}(x), \nu^{-}(x) \right) \middle| \; x \in G \right\}$$

but somewhat satisfies the counter property of

$$B = \left\{ \left(x, \mu^{+}(x), \nu^{-}(x) \right) \middle| \; x \in G \right\}$$

It is possible for an element x to be such that $\mu^{+}(x)\neq 0$ and $\nu^{-}(x)\neq 0$ when the membership function of the property overlaps that of its counter property over some portion of G. For the sake of simplicity, we shall use the symbol $B = (\mu^{+},\nu^{-})$ for the bipolar-valued fuzzy set

$$B = \left\{ \left(x, \mu^{+}(x), \nu^{-}(x) \right) \middle| \; x \in G \right\}.$$

Definition 2.2: [6]. Let X be a non-empty set with a binary operation "*" and a constant "0". Then (X,*,0) is called a BCI-algebra if it satisfies the following conditions:

(i) $((x*y)*(x*z))*(z*y) = 0$
(ii) $(x*(x*y))*y = 0$
(iii) $x*x = 0$
(iv) $x*y = 0$

and $y*x = 0$ imply $x = y$, for all $x, y, z \in X$

We can define a partial ordering $\leq$ by $x\leq y$ if and only if $x*y = 0$.

If a BCI-algebra X satisfies $0*x=0$, for all $x\in X$, then we say that X is a BCK-algebra.

A nonempty subset S of X is called a subalgebra of X if $x*y\in S$, for all $x, y\in S$.

We refer the reader to the books [5, 12] for further information regarding BCK/BCI-algebra X.

Definition 2.3: [12] Let μ be a fuzzy set in a BCK/BCI-algebra. Then μ is called a fuzzy BCK/BCI-subalgebra of X if

$$\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y\in X$.

BIPOLAR-VALUED FUZZY SUBALGEBRAS OF BCK-ALGEBRAS

From now on (X,*,0) is a BCK/BCI-algebra, unless otherwise is stated.

Definition 3.1: A bipolar-valued fuzzy set $B = (\mu^{+},\nu^{-})$ is said to be a bipolar-valued fuzzy subalgebra a BCK/BCI-algebras X if it satisfies the following conditions:

(BF1) $$\mu^{+}(x*y) \geq \min\{\mu^{+}(x),\mu^{+}(y)\}$$

(BF2) $$\nu^{-}(x*y) \leq \max\{\nu^{-}(x),\nu^{-}(y)\}$$

for all $x, y\in X$.

Example 3.2: Consider a BCI-algebra X = {0, a, b, c} with the following Cayley table:

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $B = (\mu^{+},\nu^{-})$ be a bipolar-valued fuzzy set in X with the mappings μ^{+} and ν^{-} defined by:

$$\mu^{+}(x) = \begin{cases} 0.7 & \text{if } x = 0 \\ 0.3 & \text{if } x \neq 0 \end{cases}$$

and

$$\nu^{-}(x) = \begin{cases} -0.4 & \text{if } x = 0 \\ -0.2 & \text{if } x \neq 0 \end{cases}$$

It is routine to verify that B is a bipolar-valued fuzzy subalgebra of X.

Lemma 3.3: If B is a bipolar-valued fuzzy subalgebra of X, then $\mu^{+}(0)\geq \mu^{+}(x)$ and $\nu^{-}(0)\leq\nu^{-}(x)$ for all $x\in X$.

Proposition 3.4: Let B be a bipolar-valued fuzzy subalgebra of X and let $n\in N$. Then

$$\mu^{+}(\prod^{n} x*x) \geq \mu^{+}(x)$$

(i) and $\nu^{-}(\prod^{n} x*x) \leq \nu^{-}(x)$

foranyoddnumbern

(ii) $$\mu^+(\prod^{n} x * x) = \mu^+(x)$$
$$\text{and } \nu^-(\prod^{n} x * x) = \nu^-(x)$$
for any even number n

$$\text{where} \prod^{n} x * x = \overbrace{x * x * \cdots * x}^{n-\text{times}}$$

Proof: Let $x \in X$ and assume that n is odd. Then $n = 2k-1$ for some positive integer k. We prove by induction, definition and above lemma imply that

$$\mu^+(x * x) = \mu^+(0) \geq \mu^+(x)$$

Now suppose that

$$\mu^+(\prod^{2k-1} x * x) \geq \mu^+(x)$$

Then by assumption

$$\begin{aligned}\mu^+(\prod^{2(k+1)-1} x * x) &= \mu^+(\prod^{2k+1} x * x)\\ &= \mu^+(\prod^{2k-1} x * (x * (x * x)))\\ &= \mu^+(\prod^{2k-1} x * x)\\ &\geq \mu^+(x).\end{aligned}$$

also

$$\begin{aligned}\nu^-(\prod^{2(k+1)-1} x * x) &= \nu^-(\prod^{2k+1} x * x)\\ &= \nu^-(\prod^{2k-1} x * (x * (x * x)))\\ &= \nu^-(\prod^{2k-1} x * x)\\ &\leq \nu^-(x)\end{aligned}$$

Which proves (i). Similarly we can prove (ii).

Theorem 3.5: Let B be a bipolar-valued fuzzy subalgebra of X If there exists a sequence $\{x_n\}$ in X, such that

$$\lim_{n\to\infty} \mu^+(x_n) = 1 \quad \text{and} \quad \lim_{n\to\infty} \nu^-(x_n) = -1.$$

Then $\mu^+(0) = 1$ and $\nu^-(0) = -1$.

Proof: By above lemma we have $\mu^+(0) \geq \mu^+(x)$, for all $x \in X$, thus $\mu^+(0) \geq \mu^+(x)$, for every positive integer n. Consider

$$1 \geq \mu^+(0) \geq \lim_{n\to\infty} \mu^+(x_n) = 1$$

Hence $\mu^+(0) = 1$ and similarly we have $\nu^-(0) = -1$.

Theorem 3.6: The family of bipolar-valued fuzzy subalgebras of X forms a complete distributive lattice under the ordering of bipolar-valued fuzzy set inclusion $\subset$.

Proof: Let $\{B_i \mid i \in I\}$ be a family of bipolar-valued fuzzy subalgebras of X. Since [0,1] is a completely distributive lattice with respect to the usual ordering in [0,1], it is sufficient to show that $\cap B_i = (\vee \mu_i^+, \wedge \nu_i^-)$ is a bipolar-valued fuzzy subalgebra of X. Let $x \in X$. Then

$$\begin{aligned}(\vee \mu_i^+)(x * y) &= \sup\{\mu_i^+(x * y) \mid i \in I\}\\ &\geq \sup\{\max\{\mu_i^+(x), \mu_i^+(y)\} \mid i \in I\}\\ &= \max(\sup\{\mu_i^+(x) \mid i \in I\}, \sup\{\mu_i^+(y) \mid i \in I\})\\ &= \max(\vee \mu_i^+(x), \vee \mu_i^+(y))\end{aligned}$$

Also we have

$$\begin{aligned}(\wedge \nu_i^-)(x * y) &= \inf\{\nu_i^-(x * y) \mid i \in I\}\\ &\leq \inf\{\min\{\nu_i^-(x), \nu_i^-(y)\} \mid i \in I\}\\ &= \min(\inf\{\nu_i^-(x) \mid i \in I\}, \inf\{\nu_i^-(y) \mid i \in I\})\\ &= \min(\wedge \nu_i^-(x), \wedge \nu_i^-(y))\end{aligned}$$

Hence $\cap B_i = (\vee \mu_i^+, \wedge \nu_i^-)$ is a bipolar-valued fuzzy subalgebra of X.

A fuzzy set μ of X is called anti fuzzy subalgebra of X, if $\mu(x*y) \leq \max\{\mu(x), \mu(y)\}$, for all $x, y \in X$.

Proposition 3.7: A bipolar-valued fuzzy set B of X is a bipolar-valued fuzzy subalgebra of X if and only if μ^+ is a fuzzy subalgebra and ν^- is an anti fuzzy subalgebra of X.

Definition 3.8: Let $B = (\mu^+, \nu^-)$ be a bipolar-valued fuzzy set and $(s, t) \in [-1,0] \times [0,1]$.

1) The sets $B_t^+ = \left\{x \in X \middle| \mu^+(x) \geq t\right\}$ and $B_s^- = \left\{x \in G \middle| \nu^-(x) \leq s\right\}$ which are called positive t-cut of B = (μ^+,ν^-) and negative s-cut of B = (μ^+,ν^-), respectively,
2) The sets ${}^{>}B_t^+ = \left\{x \in X \middle| \mu^+(x) > t\right\}$ and ${}^{<}B_s^- = \left\{x \in G \middle| \nu^-(x) < s\right\}$ which are called strong positive t-cut of B = (μ^+,ν^-) and the strong negative s-cut of B = (μ^+,ν^-), respectively,
3) The set $X_B^{(t,s)} = \left\{x \in X \middle| \mu^+(x) \geq t,\ \nu^-(x) \leq s\right\}$ is called an (s, t)-level subset of B,
4) The set ${}^{S}X_B^{(t,s)} = \left\{x \in X \middle| \mu^+(x) > t,\ \nu^-(x) < s\right\}$ is called a strong (s, t)-level subset of B,
5) The set of all (s, t)$\in$Im$(\mu^+)\times$Im(ν^-) is called the image of B = (μ^+,ν^-).

Theorem 3.9: Let B be a bipolar-valued fuzzy subset of X such that the least upper bound to of Im (μ^+) and the greatest lower bound s_0 of Im(ν^-) exist. Then the following conditions are equivalent:

(i) B is a bipolar-valued fuzzy subalgebra of X,
(i) For all (s, t)$\in$Im$(\nu^-)\times$Im(μ^+), the nonempty strong level subset $X_B^{(t,s)}$ of B is a (crisp) subalgebra of X.
(iii) For all (s, t)$\in$Im$(\nu^-)\times$Im$(\mu^{-+})\backslash(s_0,t_0)$, the nonempty strong level subset ${}^{S}X_B^{(t,s)}$ of B is a (crisp) subalgebra of X.
(iv) For all (s, t)$\in\lceil -1,0\rceil\times\lceil 0,1\rceil$, the nonempty strong level subset ${}^{S}X_B^{(t,s)}$ of B is a (crisp) subalgebra of X.
(v) For all (s, t)$\in$[-1,0]$\times$[0,1], the nonempty level subset $X_B^{(t,s)}$ of B is a (crisp) subalgebra of .

Proof: (i$\rightarrow$iv) Let B be a bipolar-valued fuzzy subalgebra of X, (s, t)$\in$[-1,0] $\times$[0,1] and $x,y \in^{S} X_B^{(t,s)}$.Then we have

$$\mu^+(x * y) \geq \min\left\{\mu^+(x),\ \mu^+(y)\right\} > \min\{t,t\} = t$$

and

$$\nu^-(x * y) \leq \max\left\{\nu^-(x),\ \nu^-(y)\right\} < \max\{s,s\} = s$$

Thus $x * y \in {}^{S}X_B^{(t,s)}$. Hence ${}^{S}X_B^{(t,s)}$ is a (crisp) subalgebra of X.

(iv$\rightarrow$iii) It is clear

(iii$\rightarrow$ii) Let (s, t)$\in$Im(μ+)$\times$Im(ν−). Then $X_B^{(t,s)}$ is nonempty. Since

$$X_B^{(t,s)} = \bigcap_{t>\beta,s<\alpha} {}^{S}X_B^{(\beta,\alpha)}$$

where $\beta\in$Im $(\mu^+)\backslash t_0$ and $\alpha\in$Im $(\nu^-)\backslash s_0$. Then by (iii) we get that $X_B^{(t,s)}$ is a (crisp) subalgebra of X.

(ii$\rightarrow$v) Let (s, t)$\in$[-1,0]$\times$[0,1] and $X_B^{(t,s)}$ be nonempty. Suppose that $x,y \in X_B^{(t,s)}$. Let $\alpha = \min\left\{\mu^+(x),\mu^+(y)\right\}$ and $\beta = \max\left\{\nu^-(x),\nu^-(y)\right\}$. It is clear that $\alpha\geq$s and $\beta\leq$t. Thus $x,y \in X_B^{(t,s)}$ and $\alpha\in$Im(μ^+) and $\beta\in$Im(ν^-), by (ii) $X_B^{(\alpha,\beta)}$ is a subalgebra of X, hence $x * y \in X_B^{(\alpha,\beta)}$. Then we have

$$\mu^+(x * y) \geq \min\left\{\mu^+(x),\mu^+(y)\right\}$$
$$> \min\{\alpha,\alpha\} = \alpha \geq s$$

and

$$\nu^-(x * y) \leq \max\left\{\nu^-(x),\nu^-(y)\right\}$$
$$\leq \max\{\beta,\beta\} = \beta \leq t$$

Therefore $x * y \in X_B^{(t,s)}$. Then $X_B^{(t,s)}$ is a (crisp) subalgebra of X.

(v$\rightarrow$i) Assume that the nonempty set $X_B^{(t,s)}$ is a (crisp) subalgebra of X, for any (s, t)$\in$[-1,0]$\times$[0,1]. In contrary, let $x_0, y_0\in$X be such that

$$\mu^+(x_0 * y_0) < \min\left\{\mu^+(x_0),\mu^+(y_0)\right\}$$

and

$$\nu^-(x_0 * y_0) > \max\left\{\nu^-(x_0),\nu^-(y_0)\right\}.$$

Let

$$\mu^+(x_0) = \alpha,\ \ \mu^+(y_0) = \beta,\ \ \mu^+(x_0 * y_0) = \lambda$$
$$\nu^-(x_0) = \theta,\ \nu^-(y_0) = \gamma$$

and

$$\nu^-(x_0 * y_0) = \nu$$

Then

$$\lambda < \min\{\alpha,\beta\} \qquad\qquad \nu > \max\{\theta,\gamma\}$$

put

$$\lambda_1 = \frac{1}{2}\left(\mu^+(x_0 * y_0) + \min\left\{\mu^+(x_0),\mu^+(y_0)\right\}\right)$$

and

$$\nu_1 = \frac{1}{2}\left(\nu^-(x_0 * y_0) + \max\left\{\nu^-(x_0),\nu^-(y_0)\right\}\right)$$

Therefore

$$\lambda_1 = \frac{1}{2}\left(\lambda + \min\{\alpha,\beta\}\right)$$
$$\nu_1 = \frac{1}{2}\left(\nu + \max\{\theta,\gamma\}\right).$$

Hence

$$\alpha > \lambda_1 = \frac{1}{2}\left(\lambda + \min\{\alpha,\beta\}\right) > \lambda$$
$$\nu > \nu_1 = \frac{1}{2}\left(\nu + \max\{\theta,\gamma\}\right) > \theta$$

Thus

$$\min\{\alpha,\beta\} > \lambda_1 > \lambda = \mu^+(x_0 * y_0)$$
$$\max\{\theta,\gamma\} < \nu_1 < \nu = \nu^-(x_0 * y_0)$$

so that

$$(x_0 * y_0) \notin X_B^{(\lambda_1,\nu_1)}$$

Which is a contradiction, since

$$\mu^+(x_0) = \alpha \geq \min\{\alpha,\beta\} > \lambda_1$$
$$\mu^+(y_0) = \beta \geq \min\{\alpha,\beta\} > \lambda_1$$

and

$$\nu^-(x_0) = \theta \leq \max\{\gamma,\theta\} < \nu_1$$
$$\nu^-(y_0) = \gamma \leq \max\{\gamma,\theta\} < \nu_1,$$

imply that $(x_0 * y_0) \in X_B^{(\lambda_1,\nu_1)}$. Thus

$$\mu^+(x * y) \geq \min\{\mu^+(x),\mu^+(y)\}$$

and

$$\nu^-(x * y) \leq \max\{\nu^-(x),\nu^-(y)\}$$

For all x, y$\in$X. Now the proof is completed.

Theorem 3.10: Each subalgebra of X is a level subalgebra of a bipolar-valued fuzzy subalgebra of X.

Proof: Let Y be subalgebra of X and B be a bipolar-valued fuzzy subset of X which is defined is defined by:

$$\mu^+(x) = \begin{cases} \alpha & \text{if } x \in Y \\ 0 & \text{otherwise} \end{cases}$$

$$\nu^-(x) = \begin{cases} \beta & \text{if } x \in Y \\ 0 & \text{otherwise} \end{cases}$$

Where $\alpha\in[0,1]$ and $\beta\in[-1,0]$. It is clear that $X_B^{(t,s)} = Y$. Let x,y$\in$X. We consider the following cases:

Case 1) If x, y$\in$Y, then x*y$\in$Y, therefore

$$\mu^+(x * y) = \alpha = \min\{\alpha,\alpha\} = \min\{\mu^+(x),\mu^+(y)\}$$

and

$$\nu^-(x * y) = \beta = \max\{\beta,\beta\} = \max\{\nu^-(x),\nu^-(y)\}$$

Case 2) If x, y$\notin$Y, then $\mu^+(x) = 0 = \mu^+(y)$ and $\nu^-(x) = 0 = \nu^-(y)$ and so

$$\mu^+(x * y) \geq 0 = \min\{0,0\} = \min\{\mu^+(x),\mu^+(y)\}$$

and

$$\nu^-(x * y) \leq 0 = \max\{0,0\} = \max\{\nu^-(x),\nu^-(y)\}$$

Case 3) If x$\in$Y and y$\notin$Y, then

$$\mu^+(y) = 0 = \nu^-(y), \mu^+(x) = \alpha$$

and $\nu^-(x) = \beta$. Thus

$$\mu^+(x * y) \geq 0 = \min\{\mu^+(x),\mu^+(y)\}$$

and

$$\nu^-(x * y) \leq 0 = \max\{\nu^-(x),\nu^-(y)\}$$

Case 4) If x$\notin$Y and y$\in$Y, then by the same argument as in case 3, we can conclude the results.

Therefore B is a bipolar-valued fuzzy subalgebra of X.

Theorem 3.11: Let S be a subset of X and B be a bipolar-valued subset of X which is given in the proof of Theorem 3. 10. If B is a bipolar-valued fuzzy subalgebra of X, then S is a subalgebra of X.

Proof: Let B be a bipolar-valued fuzzy subalgebra of X and x, y$\in$S. Then $\mu^+(x) = \alpha = \mu^+(y)$ and $\nu^-(x) = \beta = \nu^-(y)$, thus

$$\mu^+(x * y) \geq \min\{\mu^+(x), \mu^+(y)\} = \min\{\alpha,\alpha\} = \alpha$$

and

$$\nu^-(x * y) \leq \max\{\nu^-(x), \nu^-(y)\} = \max\{\beta,\beta\} = \beta$$

Which implies that x*y$\in$S.
Now we generalize the Theorem 3. 10

Theorem 3.12: For any chain of subalgebras of X

$$S_0 \subset S_1 \subset \cdots S_r = X$$

There exists a bipolar-valued fuzzy subalgebra B of X whose level subalgebras are exactly the subalgebras of this chain.

Proof: Consider the following sets of numbers

$$p_0 > p_1 > \cdots > p_r$$

and

$$q_0 < q_1 < \cdots < q_r$$

where each $p_i \in [0,1]$ and $q_i \in [-1,0]$. Define μ^+ and ν^- by: $\mu^+(A_i \setminus A_{i-1}) = p_i$ for all $0<i\leq r$ and $\mu^+(A_0) = p_0$ and $\nu^-(A_i \setminus A_{i-1}) = q_i$, for all $0<t\leq r$ and $\nu^-(A_0) = q_0$.

We prove that $B = (\mu^+, \nu^-)$ is a bipolar-valued fuzzy subalgebra of X. Let x, $y\in X$, we consider the following cases:

Case 1) If x, $y\in A_i\setminus A_{i-1}$, then $\mu^+(x) = p_i = \mu^+(y)$ and $\nu^-(x) = q_i = q_i = \nu^-(y)$. Since A_i is a subalgebra thus $x*y\in A_i$ so $x*y\in A_i\setminus A_{i-1}$ or $x*y\in A_{i-1}$ and in each of then we have

$$\mu^+(x*y) \geq p_i = \min\{\mu^+(x), \mu^+(y)\}$$

and

$$\nu^-(x*y) \leq q_i = \max\{\nu^-(x), \nu^-(y)\}$$

Case 2) If x, $y\in A_i\setminus A_{i-1}$ and $y\in A_i\setminus A_{i-1}$, where $i<j$. Then $\mu^+(x) = p_i = \mu^+(y) = p_j$ and $\nu^-(y) = q_j$. Since $A_j\subseteq A_i$ and A_i is a subalgebra of X, then $x*y\in A_i$. Hence

$$\mu^+(x*y) \geq p_i = \min\{\mu^+(x), \mu^+(y)\}$$

and

$$\nu^-(x*y) \leq q_i = \max\{\nu^-(x), \nu^-(y)\}$$

It is clear that

$$\mathrm{Im}(\mu^+) = \{p_0, p_1, \ldots, p_r\}$$

and

$$\mathrm{Im}(\nu^-) = \{q_0, q_1, \ldots, q_r\}$$

therefore the level subalgebras of μ^+ and ν^- are given by the chain of subalgebras

$$(\mu^+_{p_0}, \nu^-_{q_0}) \subset (\mu^+_{p_1}, \nu^-_{q_1}) \subset \cdots \subset (\mu^+_{p_r}, \nu^-_{q_r}) = X$$

We have

$$(\mu^+_{p_0}, \nu^-_{q_0}) = \left\{x \in X \,\middle|\, \mu^+(x) \geq p_0, \nu^-(x) \leq q_0\right\} = A_0$$

It is clear that $A_i \subseteq (\mu^+_{p_i}, \nu^-_{q_i})$. Let $x \in (\mu^+_{p_i}, \nu^-_{q_i})$. Then $\mu^+(x)\geq p_j$ and $\nu^-(x)\leq q_j$ then $x\notin A_j$ for $j>i$. So $\mu^+(x) \in \{p_0, p_1,\ldots,p_r\}$ and $\nu^-(x) \in \{q_0, q_1,\ldots,q_r\}$, thus $x\in A_k$ for $k\leq i$, since $A_k\subseteq A_i$ we get that $x\in A_i$. Hence $A_i = (\mu^+_{p_i}, \nu^-_{q_i})$, for $0\leq i\leq r$.

Theorem 3.13: If $B = (\mu^+, \nu^-)$ is a bipolar-valued fuzzy subalgebra of X, then the set

$$X_B = \left\{x \in X \,\middle|\, \mu^+(x) = \mu^+(0), \nu^-(0) = \nu^-(x)\right\}$$

is a subalgebra of X.

Proof: Let x, $y\in X_B$. Then $\mu^+(x) = \mu^+(0) = \mu^+(y)$ and $\nu^-(x) = \nu^-(0) = \nu^-(y)$ and so

$$\mu^+(x*y) \geq \min\{\mu^+(x), \mu^+(y)\}$$
$$= \min\{\mu^+(0), \mu^+(0)\} = \mu^+(0)$$

and

$$\nu^-(x*y) \leq \max\{\nu^-(x), \nu^-(y)\}$$
$$= \max\{\nu^-(0), \nu^-(0)\} = \nu^-(0)$$

By lemma 3.3, we get that $\mu^+(x*y) = \mu^+(0)$ and $\nu^-(x*y) = \nu^-(0)$ which means that $x*y\in X_B$.

Theorem 3.14: Let M be a subset of X. Suppose the N is a bipolar-valued fuzzy set of X defined by:

$$\mu^+_N(x) = \begin{cases} \alpha & \text{if } \alpha \in M \\ \beta & \text{otherwise} \end{cases}$$

and

$$\nu^-_N(x) = \begin{cases} \gamma & \text{if } x \in M \\ \delta & \text{otherwise} \end{cases}$$

For all $\alpha, \beta \in [0,1]$ and $\gamma, \delta \in [-1,0]$ with $\alpha\geq\beta$ and $\gamma\leq\delta$. Then N is a bipolar-valued fuzzy subalgebra if and only if M is a subalgebra of X. Moreover, in this case $X_N = M$.

Proof: Let N be a bipolar-valued fuzzy subalgebra. Let $x, y\in X$ be such that $x, y\in M$. Then

$$\mu^+_N(x*y) \geq \min\{\mu^+_N(x), \mu^+_N(y)\}$$
$$= \min\{\alpha, \alpha\} = \alpha$$

and

$$\nu_N^-(x*y) \le \max\left\{\nu_N^-(x), \nu_N^-(y)\right\}$$
$$= \min\{\gamma, \gamma\} = \gamma$$

Therefore $x*y \in M$

Conversely, suppose that M is a subalgebra of X, let $x, y \in X$.

(i) If $x, y \in M$, then $x*y \in M$, thus

$$\mu_N^+(x*y) = \alpha = \min\left\{\mu_N^+(x), \mu_N^+(y)\right\}$$

and

$$\nu_N^-(x*y) = \gamma = \max\left\{\nu_N^-(x), \nu_N^-(y)\right\}$$

(ii) If $x \notin M$ or $y \notin M$, them

$$\mu_N^+(x*y) \ge \beta = \min\left\{\mu_N^+(x), \mu_N^+(y)\right\}$$

and

$$\nu_N^-(x*y) \le \delta = \max\left\{\nu_N^-(x), \nu_N^-(y)\right\}$$

This shows that N is a bipolar-valued fuzzy subalgebra.

Moreover, we have

$$X_N := \left\{x \in X \mid \mu_N^+(x) = \mu_N^+(0), \nu_N^-(x) = \mu_N^+(0)\right\}$$
$$= \left\{x \in X \mid \mu_N^+(x) = \alpha, \nu_N^-(x) = \gamma\right\} = M$$

Definition 3.15: X is said to be Artinian if it satisfies the descending chain condition on subalgebras (simply written as DCC), that is, for every chain $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq \ldots$ of subalgebras of X, there is a natural number i such that $I_i = I_{i+1} = \ldots$.

Theorem 3.16: Each bipolar-valued fuzzy subalgebra X has finite values if and only if X is Artinian.

Proof: Suppose that each bipolar-valued fuzzy subalgebra of X has finite values. If X is not Artinian then there is a strictly descending chain

$$X = I_1 \supset I_2 \supset \ldots \supset I_n \supset \ldots$$

of subalgebras of X, where $I_i \supset I_j$ expresses $I_i \supseteq I_j$ but $I_i \ne I_j$. We now construct the bipolar-valued fuzzy set B = (μ^+, ν^-) of X by

$$\mu^+(x) := \begin{cases} \dfrac{n}{n+1} & \text{if} \quad x \in I_n \setminus I_{n+1}, n = 1,2,\ldots, \\ 1 & \text{if} \quad x \in \bigcap_{n=1}^{\infty} I_n, \end{cases}$$

$$\nu^-(x) := -\mu^+(x).$$

We first prove that B is a bipolar-valued fuzzy subalgebra of X. For this purpose, we need to verify that μ^+ is a fuzzy subalgebra of X. We assume that $x, y \in X$. Now, we consider the following cases:

Case 1: $x, y \in I_n \setminus I_{n+1}$. In this case, $x, y \in I_n$ and $x*y \in I_n$. Thus

$$\mu^+(x*y) \ge \frac{n}{n+1} = \min\left\{\mu^+(x), \mu^+(y)\right\}$$

Case 2: $x \in I_n \setminus I_{n+1}$ and $y \in I_m \setminus I_{m+1}$ $(n<m)$. In this case, $x, y \in I_n$ and $x*y \in I_n$. Thus

$$\mu^+(x*y) \ge \frac{n}{n+1} = \min\left\{\mu^+(x), \mu^+(y)\right\}$$

Case 3: $x \in I_n \setminus I_{n+1}$ and $y \in I_m \setminus I_{m+1}$ $(n>m)$. In this case, $x, y \in I_m$ and $x*y \in I_m$. Thus

$$\mu^+(x*y) \ge \frac{m}{m+1} = \min\left\{\mu^+(x), \mu^+(y)\right\}$$

Therefore μ^+ satisfies (BF1) and so μ^+ is a fuzzy subalgebra of X. This shows that B is a bipolar-valued fuzzy subalgebra of X, but the values of B are infinite, which is a contradiction. Thus X is Artinian.

Conversely, suppose that X is Artinian. If there is a bipolar-valued fuzzy subalgebra B = (μ^+, ν^-) of X with $|Im(B)| = +\infty$, then $|Im(\mu^+)| = +\infty$ or $|Im(\nu^-)| = +\infty$. Without loss of generality, we may assume that $|Im(\mu^+)| = +\infty$. Select $s_i \in |Im(\mu^+)|(i = 1,2,\ldots)$ and $s_1 < s_2 < \ldots$. Then $U(\mu^+; s_i)(i = 1,2,\ldots)$ are subalgebras of X and $U(\mu; s_1) \supseteq U(\mu+; s_2) \supseteq \ldots$ with $U(\mu^+; s_i) \ne U(\mu^+; s_{i+1})(i = 1,2,\ldots)$ which is a contradiction. Similarly for $Im(\nu^-)$. The proof is completed.

Definition 3.17: X is said to be Noetherian if every subalgebra of X is finitely generated. X is said to satisfy the ascending chain condition (briefly, ACC) if for every ascending sequence $I_1 \subseteq I_2 \subseteq \ldots$ of subalgebras of X there is a natural number n such that $I_i = I_n$, for all $i \ge n$.

Theorem 3.18: X is Noetherian if and only if for for any bipolar-valued fuzzy subalgebra B, the set Im(B) is a well ordered subset, that is, $(Im(\mu^+), \le)$ and $(Im(\nu^-), \ge)$ are well ordered subsets of [0,1] and [-1,0], respectively.

Proof: ($\Rightarrow$) Suppose that X is Noetherian. For any chain $t_1>t_2>\ldots$ of $Im(\mu^+)$, let $t_0 = \inf \{t_i \mid i=1,2,\ldots\}$. Then $I:=\{x\in G \mid \mu^+(x)>t_0\}$ is a subalgebra of X and so I is finitely generated. Let $I = (a_1,\ldots,a_k]$. Then $\mu^+(a_1)\wedge\ldots\wedge\mu^+(a_k)$ is the least element of the chain $t_1>t_2>\ldots$. Thus $(Im(\mu^+),\leq)$ is a well ordered subset of [0,1]. By using the same argument as above, we can easily show that $(Im(\nu^-),\geq)$ is a well ordered subset of [-1,0]. Therefore, Im (B) is a well ordered subset.

($\Leftarrow$) Let Im (B) be well ordered subset. If X is not Noetherian, then there is a strictly ascending sequence of subalgebras of X such that $I_1\subset I_2\subset\ldots$.
We construct the bipolar-valued fuzzy set $B = (\mu^+,\nu^-)$ of X by

$$\mu^+(x):=\begin{cases} \frac{1}{n} & \text{if} \quad x\in I_n \setminus I_{n-1}, n=1,2,\ldots, \\ 0 & \text{if} \quad a\notin \bigcup_{n=1}^{\infty} I_n, \end{cases}$$

$$\nu^-(x):=-\mu^+(x)$$

where $I_0=\phi$. By using similar method as the necessity part of Theorem 3.16, we can prove that B is a bipolar-valued fuzzy subalgebra of X. Because Im(B) is not well ordered, which is a contradiction. This completes the proof.

CONCLUSION

Bipolar-valued fuzzy set is a generalization of fuzzy sets. In the present paper, we have introduced the concept of bipolar-valued fuzzy subalgebras of BCK/BCI-algebras and investigated some of their useful properties. In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as groups, semigroups, rings, nearrings, semirings (hemirings), lattices and Lie algebras. It is our hops that this work would other foundations for further study of the theory of BCK/BCI-algebras. Our obtained results can be perhaps applied in engineering, soft computing or even in medical diagnosis [1, 9].

In our future study of fuzzy structure of BCK/BCI-algebras may be the following topics should be considered:

- To establish a bipolar-valued fuzzy ideals of BCK/BCI-algebras;
- To consider the structure of quotient BCK/BCI-algebras by using these bipolar-valued fuzzy ideals;
- To get more results in bipolar-valued fuzzy BCK/BCI-algebras and application.

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