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Escher's Tessellations in Understanding Group Theory

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Abstract: In this study, it is explained how to use Escher's tessellations in teaching group concept which is one of the most abstract concepts in mathematics. M.C.Escher's monohedral tessellations provide detailed study in an undergraduate course in abstract algebra. This study attempts to provide useful visual references for the students on learning some abstract concepts in group theory, by using Escher's tessellations. We particularly, illustrate closure, identity element, inverse element and group concept.

Key words: Abstract algebra · Teaching group theory · Symmetry groups · Escher's tessellations

INTRODUCTION

Abstract algebra is one of the most trouble courses for the students. Students think how to cope with abstract algebra concepts. Papers in literature on teaching and learning abstract algebra accept that concepts in abstract algebra have abstract nature and are difficult and propose some ways to overcome this diffucuIty [1-4].

Groups were studied not as abstract algebraic systems, but as sets of actions-what we today call groups of transformations [5]. Almost all of the topics typically encountered in an introductory group theory course can be illustrated with symmetry groups. This is an engaging approach for many students because they see a connection between two areas(group theory and geometry) and get to try out their group theory ideas in a different setting [6]. The study of the wallpaper groups and some related topics can be very effective motivation for the concepts of abstract algebra [7].

Numbers measure size, groups measure symmetry. This enigmatic sentence opens the treatise on the subject [8]. Historically, symmetry is an important factors in the beginnings of group theory. Many concepts of abstract algebra can be explored via symmetry groups. Group theory is the logic of symmetry. Therefore, symmetry groups provide a meaningful introduction to group theory and can be used to demonstrate a surprising number of topics from a standard course on groups. In particular visualization of some abstract concepts can be very effective motivation for abstract algebra. The interweaving of elementary aspects of Euclidean transformation geometry and group theory makes these groups excellent ones for study. A very specific incentive to learn about these groups is the opportunity to study examples of the imaginative interlocking patterns by the Dutch artist M. C. Escher [9].

M.C. Escher's tessellations can be studied from many points of views. We study these tessellations (tilings) mathematically by finding the underlying symmetrics. In particular, periodic tessellations can be used many concepts of abstract algebra. M.C. Escher's colored monohedral tessellations of the plane are a useful tool for exploring many concepts of abstract algebra, including groups, subgroups, cosets, conjugates, orbits and group extensions [10].

This study attempts to provide useful visual references for the students on learning some abstract concepts in group theory, by using Escher's tessellations. This has several advantages. First, the students see group theory as a useful tool. Secondly, they do not think of as a abstract course and it is very natural to finish the course with continuing applications of group theory to geometry and particularly to art. When students work with these tessellations, they can learn abstract concepts in group theory meaningfully. Also, these tessellations help students see group theory as an enjoyable topic and

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enable students to see relations between mathematics and art. It is important for students at all levels to realize and to notice mathematics in art; in nature and in science... etc. To know for students these aspects of mathematics can positively affect their attitude towards mathematics and can enable to learn mathematics more easily and meaningfully.

A tessellation is a repeating pattern of figures that covers a plane without any gaps or overlaps. Basic tessellation shapes are triangles, hexagons and rectangles. An isometry is one-to-one mapping of the plane onto itself which preserves a distance between any two points. There are three types of isometric motions in the plane, called the Euclidean isometries, namely reflection, translation, rotation. Sometimes glide-reflections are mentioned as a fourth type of isometry. It is fairly easy to see that a glide-reflection is actually a combination of a reflection and a translation. An isometry which leaves a figure invariant is called a symmetry operation.

Symmetry Groups: The totality of the symmetry operations of a tessellation constitues a group, with respect to the composition of its transformations as a group operation. This group is called the symmetry group of tessellation. The operation that combines two isometries to give another isometry is called *composition*. To begin the study of a group, we must know how the isometries (elements- in here isometries are elements of group) combine. The composition of the isometries is a two motion in the plane. The first motion acts on a tile to produce an image and then second motion acts on the image to produce a second image.

Look, for example, at the tiles of Figure 1. The pegasus tessellation has the only isometry, translations (ignoring color). The translations are the simplest isometries, in which every point of the plane moves through the same distance in the same direction.

Notice that this square in Figure 2 can generate the rest of the tessellation by simple translations.The square then tiles the plane simply by translations perpendicular to the squares edges. Choose a single tile and call it g_1. Think of the tile g_1 contained in a parallelogram and shift the tile along any of the different edges of the parallelogram.

One can see the set of motions g_1 is closed under composition and contains inverses such that if translate g_1 to another tile g_2 via translation t and then translate

Fig. 1: The pegasus tessellation

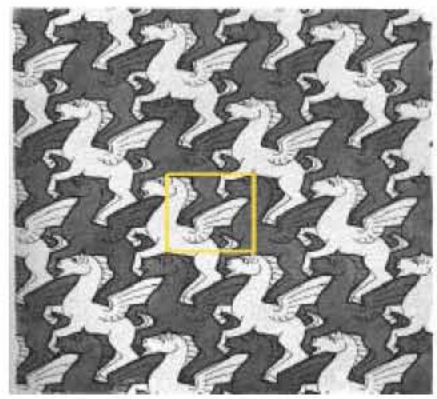

Fig. 2: Fundamental region of the pegasus tessellation

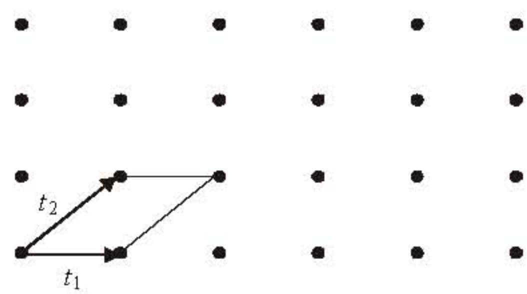

Fig. 3: A typical translation lattice

back to g_1 again via $t^{-1} = -t$. And this symmetry group has a identity elements called e, which is the isometry 'do nothing', a rigid motion that involves no motion at all. These properties mean that $G=T$ (translation group) is an infinite abelian group and the operation of composition. G have no subgroup other than $\{e\}$, identity group.

Let's look at tiles in Figure 4. The tessellation has rotational symmetry (ignoring color). Each symmetry of the figure corresponds to an isometry that preserves the figure. One symmetry, the 180° rotation, will be called r. The second object in the group of symmetries is the identity object, called e. These two isometries, e and r, are

Fig. 4: The Reptiles tessellation

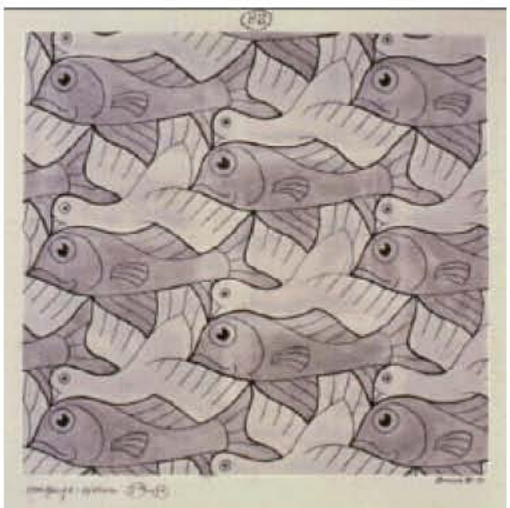

Fig. 5: The fish and the birds tessellation

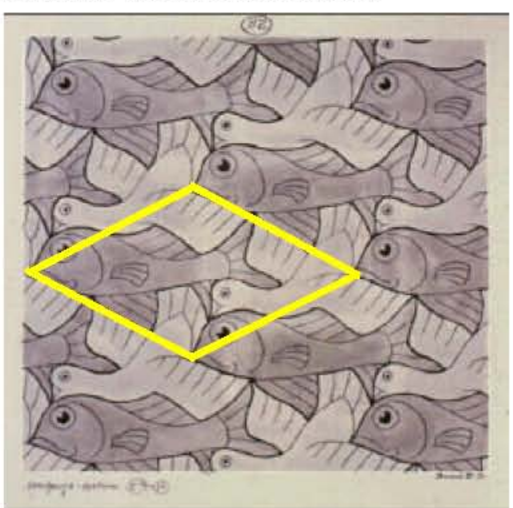

Fig. 6: The fundamental region of the fish and the birds tessellation

Table 1: Multiplication table for the symmetry group G

e	r	
e	e	r
r	r	e

the objects in the symmetry group. So, the symmetry group G has two objects, e and r. $G=\{e,r\}$ is a finite abelian group.

The operation is concisely summarized by a 'multiplication' table for the objects in the group as below Table 1.

In Figure 5, the symmetry group has two independent translations. It has no rotational nor reflective symmetry. The symmetry group is isomorphic to ZxZ. Notice that this diamond shape can generate the rest of the tessellation by simple translations.

CONCLUSION

Abstract algebra and particularly group theory is often present in undergraduate textbooks and in lectures as a summary. The concepts are abstract and axiomatic. Therefore students find difficult and complex and educators are continually revising new methods for making abstract algebra more meaningfull for the students.

Symmetry groups use the student's intution about geometry and motions to introduce group theory as a topic in a less abstract setting than it is usually encontered. The power of the abstract concept of a group is that it applies to far more than numbers and numerical operations such as addition and multiplication. The symmetry groups of tessellations are also abstract groups. The students in this method have an opportunity to see mathematics in many things involving art and architecture.

At all levels, the use computer programs can be useful to illustrate these ideas. This could enrich this course and has advantage for the students. They use easily these programs, define isometries and tessellations or plane groups are more visual for them.

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