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Differential Transform Method for Solving Boundary Value Problems Represented by System of Ordinary Differential Equations of 4th Order

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Abstract: In this paper, we obtain exact and approximate solutions for system of linear and nonlinear ordinary differential equations using differential transform method (DTM). The results show that the DTM lead to accurate results and they indicate that only a few terms lead to accurate solutions.

Key words: Differential Transform Method · Boundary Value Problem · Ordinary Differential Equation

INTRODUCTION

With the rapid development of linear and nonlinear science, many different method were proposed to solve differential equations, such as the homotopy perturbation method (HPM) [1-3] and the differential transform method (DTM) [4-10]. In this work, we aim to apply the DTM for solving boundary value problems represented by system of linear and nonlinear ordinary differential equations of 4th order. The DTM was firstly proposed by Zhou [4-6].

The Comparisons of the results of the application (DTM)s on tow applications reveal that (DTM) is very effective and convenient.

Basic Concept of Differential Transform Method (DTM): The differential transform of the *kth* derivative of function $u(x)$ is defined as follows [4-9].

$$U(k)=\frac{1}{k!}\left[\frac{d^k u}{dx^k}(x)\right]_{x=x_0} \tag{1}$$

where $u(x)$ is the original function and $U(k)$ is the transformed function.

And the differential inverse transform of $U(x)$ is defined as

$$u(x)=\sum_{k=0}^{\infty}U(k)(x-x_0)^k \tag{2}$$

The differential transform verified The following properties [4-7].

1) If $u(x) = u_1(x)\pm u_2(x)$, then $U(k) = U_1(k)\pm U_2(k)$
2) If $u(x) = cu_1(x)$, then $U(k) = cU_1(k)$, where c is a constant.
3) If $u(x)=\dfrac{d^n u_1(x)}{dx^n}$, then $U(k)=\dfrac{(k+n)!}{k!}U_1(k+n)$
4) If $u(x) = u_1(x)u_2(x)$, then $U(k)=\sum_{r=0}^{k}U_1(r)\,U_2(k-r)$
5) If $u(x) = u_1(x)u_2(x)\,.\,.\,u_n(x)$, then

$$U(k)=\sum_{r_{n-1}=0}^{k}\sum_{r_{n-2}=0}^{r_{n-1}}\cdots\sum_{r_2=0}^{r_3}\sum_{r_1=0}^{r_2}U_1(r_1)\,U_2(r_2-r_1)\cdots U_{n-1}(r_{n-1}-r_{n-2})U_n(k-r_{n-1})$$

6) If $u(x) = x^m$, then $U(k)=\delta(k-m)=\begin{cases}1 & ,k=m\\ 0 & ,k\neq m\end{cases}$
7) If $u(x)=u_1(x)\dfrac{du_2(x)}{dx}$, then

$$U(k)=\sum_{r=0}^{k}(k-r+1)U_1(r)\,U_2(k-r+1)$$

8) If $u(x) = \cos(ax+b)$, then $U(k)=\dfrac{a^k}{k!}\cos\left(\dfrac{\pi k}{2}+b\right)$

DTM for Solving Boundary Value Problems: Consider the following system of general ordinary differential equations of 4th order:

$$\begin{cases}y^{(4)}(x)=f_1(x,y,z,y',z',y'',z'',y^{(3)},z^{(3)})\\ z^{(4)}(x)=f_2(x,y,z,y',z',y'',z'',y^{(3)},z^{(3)})\end{cases},\ x\in\Omega \tag{3}$$

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Subject to the boundary conditions

$$\begin{cases} y^{(k)}(0) = \alpha_k, & y(1) = \gamma \\ z^{(k)}(0) = \beta_k, & z(1) = \mu \end{cases}$$

where:

$k = 0,1,2,$ $\qquad \alpha_k, \beta_k, \gamma, \mu = const$

And

f_1, f_2 are a nonlinear analytical functions in a general case.

Applying the DTM on (3) and by using previous properties, we get

$$\begin{cases} D\left[y^{(4)}(x)\right] = D\left[f_1(x, y, z, y', z', y'', z'', y^{(3)}, z^{(3)})\right] \\ \\ D\left[z^{(4)}(x)\right] = D\left[f_2(x, y, z, y', z', y'', z'', y^{(3)}, z^{(3)})\right] \end{cases}$$

Then

$$\begin{cases} \dfrac{(k+4)!}{k!} Y(k+4) = D\left[f_1(x, y, z, y', z', y'', z'', y^{(3)}, z^{(3)})\right] \\ \\ \dfrac{(k+4)!}{k!} Z(k+4) = D\left[f_2(x, y, z, y', z', y'', z'', y^{(3)}, z^{(3)})\right] \end{cases} \tag{4}$$

And by using DTM on the boundary conditions, we have

$$\begin{cases} y^{(k)}(0) = \alpha_k \xrightarrow{DTM} Y(k) = \dfrac{\alpha_k}{k!} \\ z^{(k)}(0) = \beta_k \xrightarrow{DTM} Z(k) = \dfrac{\beta_k}{k!} \end{cases}, \quad k = 0,1,2$$

And

$$\begin{cases} y^{(3)}(0) = a \xrightarrow{DTM} Y(3) = \dfrac{a}{6} \\ z^{(3)}(0) = b \xrightarrow{DTM} Z(k) = \dfrac{b}{6} \end{cases}$$

where *a* & *b* are unknown constants.

For $k = 0, 1,2,...$ in (4), we have the approximate solution on system (3) as follows

$$\begin{cases} y(x) = \sum_{k=0}^{\infty} Y(k)(x - x_0)^k \\ \\ z(x) = \sum_{k=0}^{\infty} Z(k)(x - x_0)^k \end{cases}$$

Numerical Examples: In this section, some examples show the usage of DTM for solving the system of linear and nonlinear boundary value problems of 4^{th} order.

Example 1: First let us consider the following linear system of ordinary differential equations

$$\begin{cases} y^{(4)}(x) - xy'(x) + xz''(x) - z'(x) = x \\ z^{(4)}(x) + xz^{(3)}(x) - xy^{(3)}(x) - y''(x) = 0 \\ \qquad 0 \le x \le 1 \end{cases} \tag{5}$$

Subject to the boundary conditions

$$\begin{cases} y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1, \quad y(1) = e - 1 \\ z(0) = 1, \quad z'(0) = 1, \quad z''(0) = 1, \quad z(1) = e \end{cases} \tag{6}$$

The exact solution of this system are:

$y(x) = e^x - x, \qquad z(x) = e^x$

Applying the DTM choosing $x_0 = 0$, then (5) is transformed in the following form:

$$\begin{cases} \dfrac{(k+4)!}{k!} Y(k+4) = \delta(k-1) + \left(\displaystyle\sum_{m=0}^{k} (k-m+1)\delta(m-1) Y(k-m+1) \right) \\ \quad - \left(\displaystyle\sum_{m=0}^{k} (k-m+1)(k-m+2)\delta(m-1) Z(k-m+2) \right) + (k+1) Z(k+1) \\ \\ \dfrac{(k+4)!}{k!} Z(k+4) = (k+2)(k+1) Y(k+2) \\ \quad - \left(\displaystyle\sum_{m=0}^{k} (k-m+3)(k-m+2)(k-m+1)\delta(m-1) Z(k-m+3) \right) \\ \quad + \left(\displaystyle\sum_{m=0}^{k} (k-m+3)(k-m+2)(k-m+1)\delta(m-1) Y(k-m+3) \right) \end{cases}$$

$$\begin{cases} y(0) = 1 \to Y(0) = 1, \qquad z(0) = 1 \to Z(0) = 1 \\ y'(0) = 0 \to Y(1) = 0, \qquad z'(0) = 1 \to Z(0) = 1 \\ y''(0) = 1 \to Y(2) = \dfrac{1}{2}, \qquad z''(0) = 1 \to Z(2) = \dfrac{1}{2} \\ y^{(3)}(0) = a \to Y(3) = \dfrac{a}{6}, \qquad z^{(3)}(0) = b \to Z(3) = \dfrac{b}{6} \end{cases}$$

$$k=0 \Rightarrow Y(4)=\frac{1}{24}, \quad Z(4)=\frac{1}{24}$$

$$k=1 \Rightarrow Y(5)=\frac{1}{120}, \quad Z(5)=-\frac{b}{120}+\frac{a}{60}$$

$$k=2 \Rightarrow Y(6)=\frac{1}{360}-\frac{b}{720}, \quad Z(6)=\frac{1}{720}$$

Then, the approximate solutions of (5) subject to (6) are given by

$$\begin{cases} y(x)=\sum_{k=0}^{\infty} Y(k)\, x^k \\ z(x)=\sum_{k=0}^{\infty} Z(k)\, x^k \end{cases}$$

By the boundary conditions $y(1) = e - 1$ & $z(1) = e$ and Maple program for $n = 6$, we have

$$a=1.009690971, \quad b=1.009180920$$

For $n = 7$, we have

$$a=1.001368346, \quad b=1.001285056$$

In Table. 1 We give the comparison of the errors of (5) for $y(x)$ by truncating six terms $y_0 + y_1 + \ldots\ y_5$ and seven terms $y_0 + y_1 + \ldots\ y_6$ of the solution series by using Maple.

In Table 2 we give the comparison of the errors of (5) for $z(x)$ by truncating six terms $z_0 + z_1 + \ldots\ z_5$ and seven terms $z_0 + z_1 + \ldots\ z_6$ of the solution series by using Maple.

Table 1: Numerical results for example 1

x	Error of DTM n=6	Error of DTM n=7
0	0.00000000000	0.00000000000
0.1	1.6138285 e -06	2.2811140 e -07
0.2	1.2829961 e -05	1.8219020 e -06
0.3	4.2551370 e -05	6.1107549 e -06
0.4	9.7339020 e -05	1.4245939 e -05
0.5	1.7854089 e -04	2.6826375 e -05
0.6	2.7807496 e -04	4.3177178 e -05
0.7	3.7271018 e -04	6.0124842 e -05
0.8	4.1670153 e -04	7.0125191 e -05
0.9	3.3259200 e -04	5.8556984 e -05
1	4.5149632 e -06	2.3144653 e -07

Table 2: Numerical results for example 1

x	Error of DTM n=6	Error of DTM n=7
0	0.00000000000	0.00000000000
0.1	1.5296701 e -06	2.1435255 e -07
0.2	1.2177096 e -05	1.7148353 e -06
0.3	4.0462711 e -05	5.7666486 e -06
0.4	9.2768968 e -05	1.3488692 e -05
0.5	1.7057135 e -04	2.5497093 e -05
0.6	2.6632338 e -04	4.1202680 e -05
0.7	3.5783964 e -04	5.7606579 e -05
0.8	4.0103276 e -04	6.7449619 e -05
0.9	3.2081748 e -04	5.6528936 e -05
1	1.2241896 e -06	3.5564214 e -07

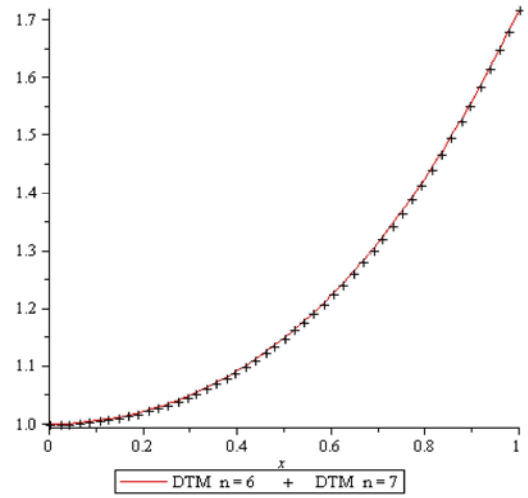

Fig. 1: The approximate solution for $y(x)$ by using DTM for example 1

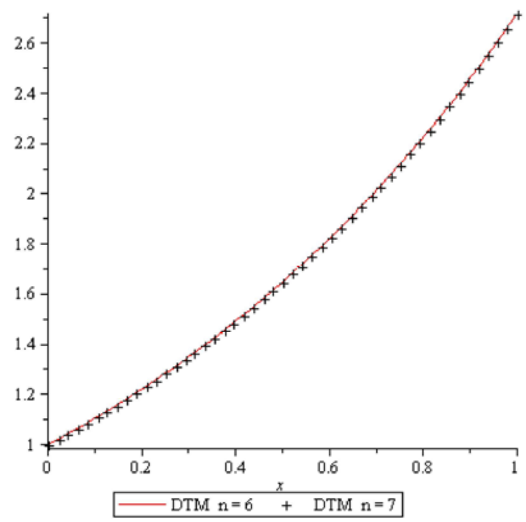

Fig. 2: The approximate solution for $z(x)$ by using DTM for example 1

The following graphs show the errors for example 1 by taking n=6 and n=7 are drawn in Fig. 3, Fig. 4, Fig. 5 and Fig. 6 respectively.

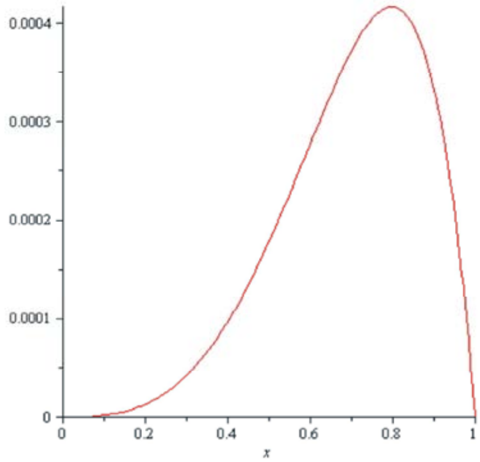

Fig. 3: The error of approximate solution for $y(x)$ for example 1 by taking n = 6

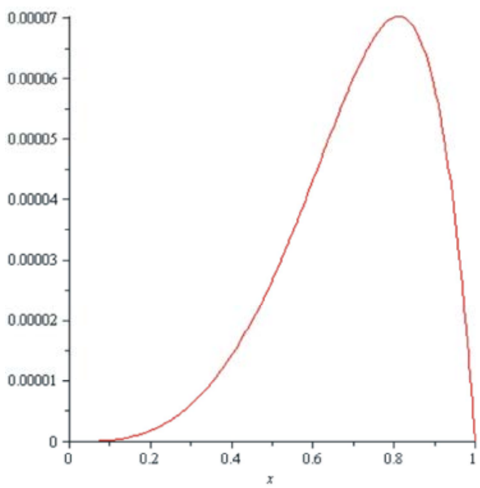

Fig. 4: The error of approximate solution for y (x) for example 1 by taking n = 7

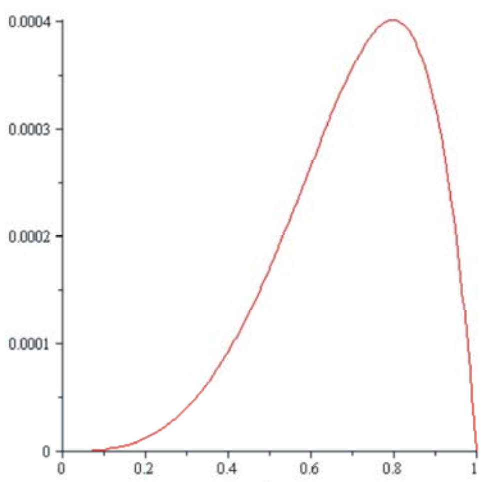

Fig. 5: The error of approximate solution for $z(x)$ for example 1 by taking n = 6

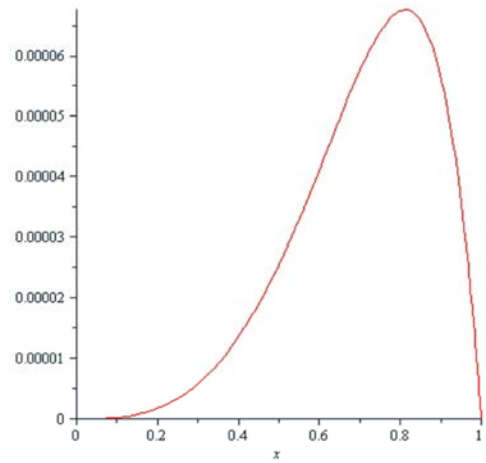

Fig. 6: The error of approximate solution for $z(x)$ for example 1 by taking n = 7

Example 2: Considering the following nonlinear system of ordinary differential equations

$$\begin{cases} y^{(4)}(x) - y''(x) - y^{(3)}(x)z'(x) = 24x\sin(x) - 12x^2 + 26 \\ z^{(4)}(x) - 24xz''(x) - y^{(3)}(x)z(x) = \cos(x) \\ \qquad 0 \le x \le 1 \end{cases} \tag{7}$$

Subject to the boundary conditions

$$\begin{cases} y(0) = 0, \quad y'(0) = 0, \quad y''(0) = -2, \quad y(1) = 0 \\ z(0) = 1, \quad z'(0) = 0, \quad z''(0) = -1, \quad z(1) = \cos(1) \end{cases} \tag{8}$$

The exact solution of this system are:

$y(x) = -x^2 + x^4, \qquad z(x) = \cos x$

Applying the DTM choosing x_0 = 0, then (7) is transformed in the following form:

$$\begin{cases} \dfrac{(k+4)!}{k!}Y(k+4) = (k+2)(k+1)Y(k+2) \\ \quad + \left(\displaystyle\sum_{m=0}^{k}(m+3)(m+2)(m+1)(k-m+1)Y(m+3)Z(k-m+1)\right) \\ \quad + 26\delta(k) - 12\delta(k-2) + 24\left(\displaystyle\sum_{m=0}^{k}\frac{\delta(m-1)}{(k-m)!}\sin\left(\frac{\pi}{2}(k-m)\right)\right) \\ \\ \dfrac{(k+4)!}{k!}Z(k+4) = \dfrac{1}{k!}\cos\left(\dfrac{\pi}{2}k\right) \\ \quad + 24\left(\displaystyle\sum_{m=0}^{k}(k-m+2)(k-m+1)\delta(m-1)Z(k-m+2)\right) \\ \quad + \left(\displaystyle\sum_{m=0}^{k}(m+3)(m+2)(m+1)Y(m+3)Z(k-m)\right) \end{cases}$$

$$\begin{cases} y(0)=0\rightarrow Y(0)=0, & z(0)=1\rightarrow Z(0)=1 \\ y'(0)=0\rightarrow Y(1)=0, & z'(0)=0\rightarrow Z(0)=0 \\ y''(0)=-2\rightarrow Y(2)=-1, & z''(0)=-1\rightarrow Z(2)=\dfrac{-1}{2} \\ y^{(3)}(0)=a\rightarrow Y(3)=\dfrac{a}{6}, & z^{(3)}(0)=b\rightarrow Z(3)=\dfrac{b}{6} \end{cases}$$

$$k=0\Rightarrow Y(4)=1,\quad Z(4)=\frac{a+1}{24}$$

Then, the approximate solutions of (7) subject to (8) are given by

$$\begin{cases} y(x)=\sum_{k=0}^{\infty} Y(k)\,x^k \\ z(x)=\sum_{k=0}^{\infty} Z(k)\,x^k \end{cases}$$

By the boundary condition $y(1) = 0$ & $z(x) = \text{cost}(1)$ and Maple program for $n = 5$, we have

$$a=0,\quad b=-\frac{13}{4}+6\cos(1)$$

Then we obtain the exact solution for $y(x)$ as follows

$$y(x)=-x^2+x^4$$

In Table. 3 we give the errors of (7) for $z(x)$ by truncating five terms $z_0 + z_1 + ...+ z_4$ of the solution series by using Maple.

The following graph show the errors for example 2 by taking n=5 is drawn in Fig. 8

Table 3: Numerical results for example 2

x	Error of DTM n=5
0	0.00000000000
0.1	1.3629942 e -06
0.2	1.0826020 e -05
0.3	3.5826843 e -05
0.4	8.1646426 e -05
0.5	1.4894034 e -04
0.6	2.3031684 e -04
0.7	3.0599640 e -04
0.8	3.3859538 e -04
0.9	2.6708735 e -04
1	1.04862485 e -06

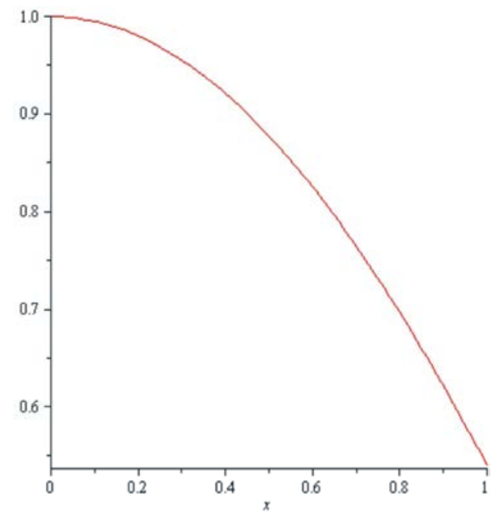

Fig. 7: The approximate solution for $z(x)$ by using DTM for example 2

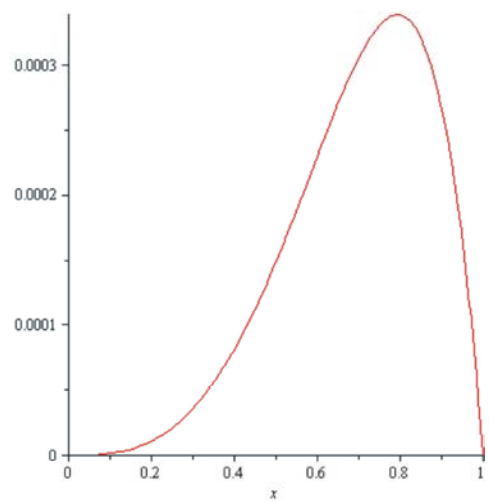

Fig. 8: The error of approximate solution for $z(x)$ for example 2 by taking $n = 5$

CONCLUSION

In this paper, we have used the differential transform method (DTM) for solving linear and nonlinear boundary value problems represented by system of ordinary differential equations of 4th order.

The DTM is very powerful and efficient in finding the exact and approximate solutions for a wide class of boundary value problems. This method gives more realistic series solutions that converge very rapidly. Thus, we conclude that the DTM technique can be considered as an efficient method for solving linear and nonlinear problems.

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