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Homotopy Perturbation Method in Solving Nonlinear Boundary Value Problems

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Abstract: In this paper, we present a modification to homotopy perturbation method for solving nonlinear Boundary value problems., Some examples are given, revealing its effectiveness and convenience. Also the results obtained by modified homopy perturbation method (MHPM) is superior to that obtained by the original homotopy perturbation method

Key words: Homotopy Perturbation Method · Boundary Value Problem · Nonlinear Ordinary Differential Equation

INTRODUCTION

A new perturbation method called homotopy perturbation method was proposed by Ji Huan He in 1997 and systemically description in 2000 [1-3] which is, in fact, a coupling of the traditional perturbation method and homotopy in topology. This method was applied for solving many mathematical models.

We modified this method for solving several forms of nonlinearity (i.e. nonlinear polynomials, trigonometric nonlinearity, hyperbolic nonlinearity, exponential nonlinearity and logarithmic nonlinearity) that given in nonlinear ordinary differential equations. We modified this method by introducing the homotopy parameter " p" (small parameter) in nonlinear terms by multiply in the unknown function u , then we are expanding this(these) term(s) by Taylor's series and by substituting the solve of HPM and by equating the coefficients of like power of parameter homotopy p , we obtain u_0, u_1, u_2.

Finally, the approximate solution is given by;

$$v = \lim_{p\to 1} u = u_0 + u_1 + u_2 + ...$$

Basic Idea of Homotopy Perturbation Method: Consider a nonlinear a boundary value problem [4-7].

$$A(u) - f(u(r)) = 0, \quad r \in \Omega \tag{1}$$

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma$$

where A is a general differential operator, B a boundary operator, $f(u)$ is a known nonlinear analytic function on a Hilbert space, Γ is the boundary of domain Ω, the operator A can be divided into two parts: a linear part A and a nonlinear part N. Therefore Eq. (1) can be rewritten as follows:

$$L(u) + N(u) = f(u) \tag{2}$$

The homotopy on Eq. (2) can be constructed by;

$$H(v, p) = (1-p)\,[L(v) - L(u_0)] + p\,[L(v) + N(v) - f(v)] = 0$$

Or

$$H(v, p) = L(v) - L(u_0) + p[L(u_0) + N(v) - f(v)] = 0 \tag{3}$$

where $p \in [0,1]$ is an Embedding parameter (Homotopy Parameter), u_0 is an initial solution of Eq. (1).

Obviously, from Eq. (3) we will have

$$H(v, 0) = L(v) - L(u_0) = 0$$

$$H(v, 1) = A(v) - f(v) = 0 \tag{4}$$

If the embedding parameter p is considered as a "small parameter" and by applying the classical perturbation technique, so we can assume that the solution of Eq. (3), can be given by a power series in p, i.e.

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$$v = v_0 + pv_1 + p^2 v_2 + ... \quad (5)$$

Then, the approximate solution of Eq. (1) as;

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots \quad (6)$$

A New Modified Homotopy Perturbation Method: We implement the modified homotopy perturbation method as follows:

Replacing the unknown function v by pv in nonlinear terms f(v), then expanding these terms by Taylor's series around v0 and by substituting Eq. (5) into the both Taylor's series and Eq. (4) and finally by equating the coefficients of like power of parameter homotopy p , then we obtain u_0, u_1, u_2,...

We explain our modified method by considering two examples in the following.

Implementation of the Method

Example 1: Consider the two points nonlinear boundary value problem.

$$u'' - 2\cos u = 0 \quad (7)$$

$$u'(0) = 0, \quad u(0) = 0$$

We rewrite Eq.(7) by;

$$u'' - 2\cos(pu) = 0 \quad (8)$$

We expend cos (pu) by Taylor's series around $u_0 = 0$, i.e.

$$\cos(pu) = 1 - \frac{p^2u^2}{2} + \frac{p^4u^4}{24} + ... \quad (9)$$

According to the HPM , then the solution of Eq. (8) can be written in the following form.

$$u = u_0 + pu_1 + p^2u_2 + ... \quad (10)$$

And by substituting Eq. (9) and Eq. (10) into Eq. (8), we have,

$$u_0'' + pu_1'' + p^2u_2'' + ... = 2[1 - \frac{p^2}{2}(u_0 + pu_1 + p^2u_2 + ...)^2 + \frac{p^4}{24}(u_0 + pu_1 + p^2u_2 + ...)^4 + ...]$$

And equating the coefficients of p with the same power leads to,

$$p^0 : u_0'' = 2; \quad u_0(0) = 0, \quad u_0'(0) = 0 \quad (11)$$

$$p^1 : u_1'' = 0; \quad u_1(0) = 0, \quad u_1'(0) = 0 \quad (12)$$

$$p^2 : u_2'' = -u_0^2; \quad u_2(0) = 0, \quad u_2'(0) = 0 \quad (13)$$

$$p^3 : u_3'' = -2u_0u_1; \quad u_3(0) = 0, \quad u_3'(0) = 0 \quad (14)$$

$$p^4 : u_4'' = -u_1^2 - 2u_0u_2 + \frac{1}{12}u_0^4; \quad u_4(0) = 0, \quad u_4'(0) = 0 \quad (15)$$

Therefore, the solutions of the above problems can be expressed as;

$$u_4 = \frac{x^{10}}{600}\ u_3 = 0, \ u_2 = -\frac{x^6}{30}, \quad u_1 = 0, \quad u_0 = x^2$$

Then the fifth term approximate solution of BVP (7) is given by;

$$v = \lim_{p\to 1} u = u_0 + u_1 + u_2 + ... = x^2 - \frac{x^6}{30} + \frac{x^{10}}{600}$$

In other hand the application of the original HPM leads to the following fifth term approximate solution of BVP (7):

$$u_4 = 0 \ \ u_3 = -\frac{x^6}{30}, \quad u_2 = 0, \quad u_1 = x^2, \quad u_0 = 0$$

Table 1: Comparison between the errors obtained by original HPM against MHPM for different values of variable x in the interval [0,1]

x	HPM	MHPM
0	0.00000000	0.0000000
0.1	1×10^{-9}	1×10^{-9}
0.2	3.84×10^{-7}	0.0000000
0.3	9.834×10^{-6}	1×10^{-8}
0.4	9.8053×10^{-5}	3.07×10^{-7}
0.5	5.82293×10^{-4}	4.449×10^{-6}
0.6	2.487097×10^{-3}	3.9398×10^{-5}
0.7	8.443383×10^{-3}	2.47809×10^{-4}
0.8	2.4166886×10^{-2}	1.210192×10^{-3}
0.9	6.0540218×10^{-2}	4.857585×10^{-3}
1	1.3609201×10^{-1}	1.665289×10^{-2}

Example 2: Consider the following nonlinear boundary value problem.

$$u'' - shu = 0 \quad (16)$$

$$u(1) = 0, \quad u'(0) = 0, \quad u(0) = 0$$

We rewrite Eq. (16) by;

$$u''' = sh\ (pu) \quad (17)$$

We write $sh(pu)$ by Taylor's series around zero by;

$$sh(pu) = pu + \frac{p^3u^3}{6} + \frac{p^5u^5}{120} + \ldots \quad (18)$$

By HPM, then the solve of Eq. (17) is given by;

$$u = u_0 + pu_1 + p^2u_2 + \ldots \quad (19)$$

And by substituting Eq. (18) and Eq. (19) into Eq. (17), we have,

$$u_0''' + pu_1''' + p^2u_2''' + \ldots = p(u_0 + pu_1 + p^2u_2 + \ldots) + \frac{p^3}{6}(u_0 + pu_1 + p^2u_2 + \ldots)^3$$
$$+ \frac{p^5}{120}(u_0 + pu_1 + p^2u_2 + \ldots)^5 + \ldots$$

By equating the coefficients of like power of parameter homotopy p, we have;

$$p^0 : u_0''' = 0; \quad u_0(0) = 0 \quad u_0'(0) = 0 \quad u_0''(0) = \alpha \quad (20)$$

$$p^1 : u_1''' = u_0; \quad u_1(0) = 0 \quad u_1'(0) = 0 \quad u_1''(0) = 0 \quad (21)$$

$$p^2 : u_2''' = u_1; \quad u_2(0) = 0 \quad u_2'(0) = 0 \quad u_2''(0) = 0 \quad (22)$$

$$p^3 : u_3''' = u_2 + \frac{1}{6}u_0^3; \quad u_3(0) = 0 \quad u_3'(0) = 0 \quad u_3''(0) = 0 \quad (23)$$

$$p^4 : u_4''' = u_3 + \frac{1}{2}u_0^2u_2; \quad u_4(0) = 0 \quad u_4'(0) = 0 \quad u_4''(0) = 0 \quad (24)$$

$$p^5 : u_4''' = u_4 + \frac{1}{2}u_0u_1^2 + \frac{1}{2}u_0^2u_2 + \frac{1}{120}u_0^5; \quad u_5(0) = 0, u_5'(0) = 0, u_5''(0) = 0 \quad (25)$$

From Eq. (20), we have,

$$u_0 = x + \frac{\alpha}{2}x^2$$

From Eq. (21), we have,

$$u_1 = \frac{1}{24}x^4 + \frac{\alpha}{120}x^5$$

From Eq. (22), we have,

$$u_2 = \frac{1}{5040}x^7 + \frac{\alpha}{40320}x^8$$

From Eq. (23), we have,

$$u_3 = \frac{1}{720}x^6 + \frac{\alpha}{840}x^7 + \frac{\alpha^2}{2688}x^8 + \frac{\alpha^3}{24192}x^9 + \frac{1}{3628800}x^{10} + \frac{\alpha}{39916800}x^{11}$$

From Eq. (24), we have,

$$u_4 = \frac{1}{22680}x^9 + \frac{11\alpha}{302400}x^{10} + \frac{131\alpha^2}{13305600}x^{11} + \frac{131\alpha^3}{159667200}x^{12}$$
$$+ \frac{1}{6227020800}x^{13} + \frac{\alpha}{87178291200}x^{14}$$

From Eq. (25), we have,

$$u_5 = \frac{1}{40320}x^8 + \frac{\alpha}{24198}x^9 + \frac{\alpha^2}{34560}x^{10} + \frac{\alpha^3}{95040}x^{11}$$
$$+ \frac{1}{12}\left(\frac{367}{39916800} + \frac{\alpha^4}{42240}\right)x^{12} + \frac{1}{12}\left(\frac{367}{39916800} + \frac{\alpha^4}{42240}\right)x^{12}$$
$$+ \frac{1}{13}\left(\frac{281\alpha}{39916800} + \frac{\alpha^5}{506880}\right)x^{13} + \frac{1699\alpha^2}{14529715200}x^{14} + \frac{1699\alpha^3}{217945728000}x^{15}$$
$$+ \frac{1}{20299789888000}x^{16} + \frac{\alpha}{355687428096000}x^{17}$$

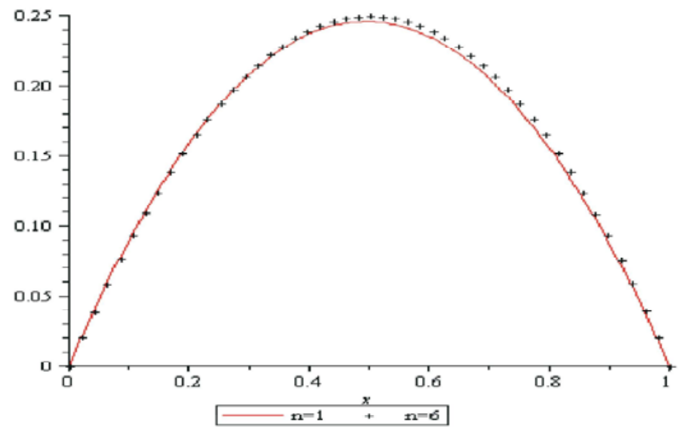
Then, the approximate solution of Eq. (16) with boundary conditions is given by,

$$v = \lim_{p \to 1} u = u_0 + u_1 + u_2 + \ldots$$

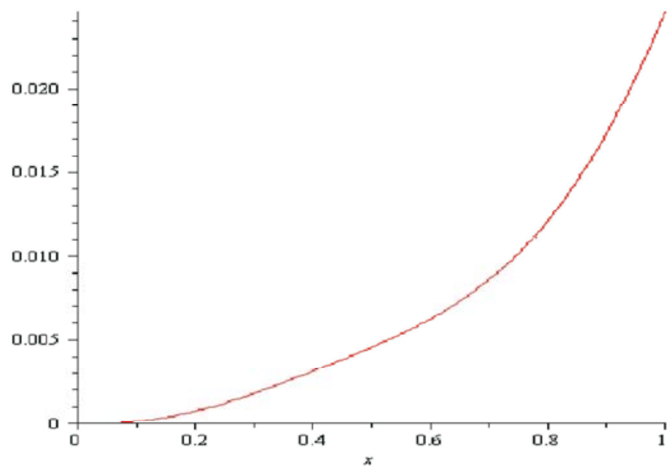
By the boundary condition $u(1) = 0$ and Maple Program, We have,

$$\alpha = -2.0499785918$$

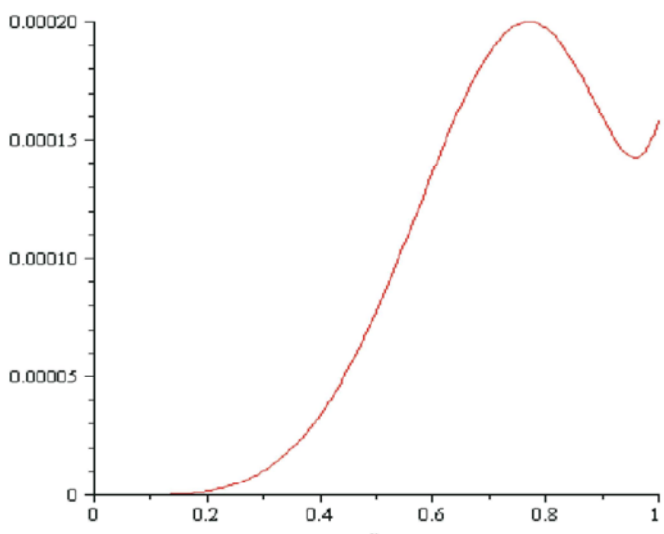
The figure illustrates the solution for $n = 1$ and $n = 6$ in [0,1],



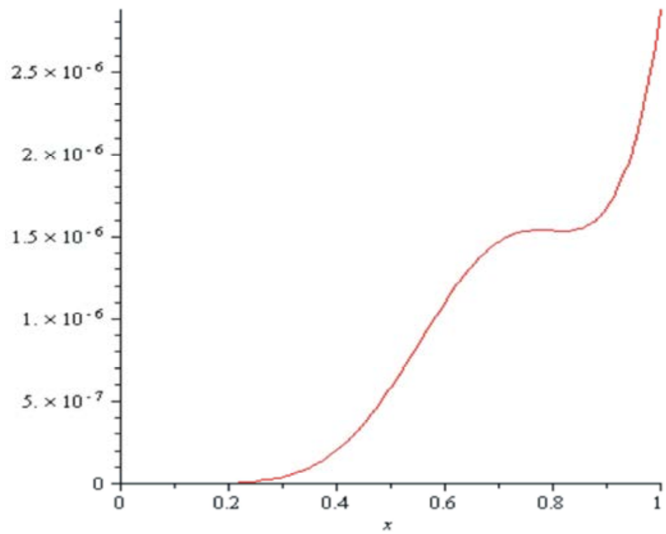
The figure illustrates the Error for $n = 2$



The figure illustrates the Error for $n = 4$



The figure illustrates the Error for $n = 6$



The table illustrates the Error for n = 2, n = 4 and n = 6

x	Error	Error	Error
0	0.000000	0.0000000	0.00000
0.1	1.245737×10^{-4}	6.58185×10^{-8}	3×10^{-11}
0.2	7.32795×10^{-4}	1.6829×10^{-6}	3.1×10^{-9}
0.3	1.80008×10^{-3}	1.02376×10^{-5}	3.94×10^{-8}
0.4	3.11556×10^{-3}	3.3868×10^{-5}	2.025×10^{-7}
0.5	4.55689×10^{-3}	7.7888×10^{-5}	5.827×10^{-7}
0.6	6.24799×10^{-3}	1.3675×10^{-4}	1.0998×10^{-6}
0.7	8.57448×10^{-3}	1.8696×10^{-4}	1.4727×10^{-6}
0.8	1.21188×10^{-2}	1.9753×10^{-4}	1.5365×10^{-6}
0.9	1.73649×10^{-2}	1.6067×10^{-4}	1.67176×10^{-6}
1	2.45939×10^{-2}	1.5856×10^{-4}	2.876173×10^{-6}

CONCLUSION

Modified homotopy perturbation method is applied to the nonlinear boundary value problems. Comparison of the result obtained by the present method with that obtained by original homotopy perturbation method reveals that the present method is very effective and convenient.

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