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A Transnormal Partial Tube Around A Non-Transnormal Manifold

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Abstract: In this paper we study transnormal partial tubes. The main aim is to introduce an example of a transnormal partial tube whose base is not transnormal. Our example will be of a special type of embeddings in $\mathbb{R}^6$.

Key words: Transnormal manifold · Partial tube · Generating frame

INTRODUCTION

The idea of transnormality is a generalization of the concept of an m-hypersurface of constant width in $\mathbb{R}^{m+1}$ it is due to S. Robertson [1, 2, 3] and contributions have been made by B.Wegner [4-7], S.Carter and K. Al-Banawi [8-12]. The notion of constant width can be formulated as follows. Let M be a smooth compact connected m-manifold without boundary that is smoothly embedded in $\mathbb{R}^{m+1}$. A chord of M is normal if it is normal to M at one of its endpoints and binormal if it is normal to M at both end points.The manifold M is of constant width if and only if every normal chord of M is binormal to M. Each point of the endpoints is called the opposite of the other.

Let M be a smooth connected m-manifold without boundary and let f: $M \to \mathbb{R}^n$ be a smooth embedding of M into $\mathbb{R}^n$. Let $V=f(M)$. For each point $p \in V$ there exists a unique tangent plane T_pV tangent to V at p with dimension m and a unique normal plane N_pV normal to V at p with dimension n-m. Thus, there are maps T and N with $T(p) = T_pV$ and $N(p) = N_pV$.

Definition 1: [3] The m-manifold V is *transnormal* in $\mathbb{R}^n$ iff

$\forall p, q \in V$ if $q \in N(p)$ then $N(q) = N(p)$.

Let W be the space of normal planes of V, say $W = N(V)$. S. Robertson showed that for any transnormal embedding V in $\mathbb{R}^n$ the order of N as a covering map is always finite [1]. If V is transnormal in $\mathbb{R}^n$ and the order of N is r, then V is called an r-transnormal manifold.

Definition 2: [3] Let V be a transnormal manifold in $\mathbb{R}^n$. Then the *generating frame* of V at p is;

$\phi(p) = V \cap N(p)$

If V is r-transnormal, then $|\phi(p)| = r$ where $| \dots |$ is the cardinality.

It is true that any two generating frames are isometric. That is, if $\phi(p_1)$ and $\phi(p_2)$ are generating frames, then there exists a map F: $\phi(p_1) \to \phi(p_2)$ which preserves distance. Also if V is a compact r-transnormal manifold, then r is even [2].

Transnormal Spherical Partial Tubes: The general definition of a *partial tube* was introduced in [13] as follows. Let M be a smooth connected m-manifold without boundary. Let $f: M \to \mathbb{R}^n$ be a smooth embedding of M into the Euclidean space $\mathbb{R}^n$, $n = m + k$. For $p \in M$, let T_pM be the tangent plane of M at p. Consider the normal bundle of M $\aleph = \{(p,v) : p \in M, v \perp T_pM\}$ and the smooth endpoint map $\eta : \aleph \to \mathbb{R}^n$ defined by $\eta(p, v) = p + v$.

Let Σ be the set of singular points of η. Let $P \subset \aleph$ be a smooth subbundle with type fibre S such that S is a smooth submanifold of $\mathbb{R}^k$. If $P \cap \Sigma$ is empty, then P is a smooth manifold and η/P is a smooth embedding called a *partial tube* around f. The manifold $V=f(M)$ is usually called the *base* of the partial tube h. A partial tube is *spherical* if S is a sphere. The word partial is used if S is embedded in a proper subplane of the normal plane at p. Otherwise; the spherical tube is called a *full tube*. Embeddings similar to h with S being an image of an embedding were studied in [13].

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Assume that $f : M \to \mathbb{R}^n$ is a smooth r-transnormal embedding of the compact connected m-manifold M without boundary in the Euclidean space $\mathbb{R}^n$. Then the next theorem ensures the existence of $\xi \succ 0$ such that the above full tube is the image of an embedding. Also if $p \in V = f(M)$, then the normal plane of V at p, $N(p)$, intersects the full tube at points based at the points of the generating frame $\phi(p)$. By a similar argument this result holds if the normal bundle is replaced by a subbundle P of $\aleph$.

Theorem 1: [11] Let $f : M \to \mathbb{R}^n$ be a smooth r-transnormal embedding of the compact connected m-manifold M without boundary into the Euclidean space $\mathbb{R}^n$. Then for some $\xi \succ 0$ sufficiently small.

- The map $\eta| \aleph^{\xi} V$ is an embedding and
- For all $p \in V$, for all $(q, v) \in \eta| \aleph^{\xi} V$,

$\eta(q, v) \in N(p)$ iff $q \in N(p) \cap V$.

In the next theorem the dimension of the normal plane is the sum of the dimension of the parallel normal plane (d) and the dimension of its compliment (k).

Theorem 2: [9] Let f: $M \to \mathbb{R}^{m+d+k}$ be a smooth r-transnormal embedding of the compact connected m-manifold M without boundary into the Euclidean space $\mathbb{R}^{m+d+k}$. Then there exists a $2r$-transnormal embedding of a $(k-1)$ -sphere bundle over $V=f(M)$ in $\mathbb{R}^{m+d+k}$ with image a partial tube and V as its base.

A Transnormal Partial Tube around a Non-Transnormal Manifold: This section is an example of a transnormal partial tube with a base that is not transnormal.

Let M be the set of 3×3 real symmetric matrices. For $A, B \in M$, define the metric,

$$<A,B> = \text{trace } A B$$

So if $A = \begin{pmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_4 & b_5 \\ b_4 & b_2 & b_6 \\ b_5 & b_6 & b_3 \end{pmatrix}$

then $< A, B> = a_1b_1 + a_2b_2 + a_3b_3 + 2(a_4b_4 + a_5b_5 + a_6b_6)$.

Now assume that $\mathbb{R}^6$ is identified with M such that $x = \{x_1,......,x_6\} \in \mathbb{R}^6$ is represented in M by the matrix.

$$\begin{pmatrix} x_1 & \frac{1}{\sqrt{2}}x_4 & \frac{1}{\sqrt{2}}x_5 \\ \frac{1}{\sqrt{2}}x_4 & x_2 & \frac{1}{\sqrt{2}}x_6 \\ \frac{1}{\sqrt{2}}x_5 & \frac{1}{\sqrt{2}}x_6 & x_3 \end{pmatrix}$$

Consider the set

$S = \{A \in M$: $traceA = 0$, $traceA^2 = 1\}$
$= \{x \in \mathbb{R}^6$: $x_1 + x_2 + x_3 = 0$, $\|x\| = 1)$..

The set S is the unit 4- sphere in the 5- plane $\{x \in \mathbb{R}^6 : x_1 + x_2 + x_3 = 0$.

Consider the embedding f of the projective plane P^2 in $\mathbb{R}^6$ given by;

$$f(x,y,z) = \left(\sqrt{\frac{3}{2}}(x^2-\frac{1}{3}), \sqrt{\frac{3}{2}}(y^2-\frac{1}{3}), \sqrt{\frac{3}{2}}(z^2-\frac{1}{3}), \sqrt{3}xy, \sqrt{3}xz, \sqrt{3}yz\right)$$

where $x^2 + y^2 + z^2 = 1$. A point on $f(P^2)$ is represented by $A_{(x, y, z)}$ where

$$A_{(x,y,z)} = \sqrt{\frac{3}{2}}\begin{pmatrix} x^2-\frac{1}{3} & xy & xz \\ xy & y^2-\frac{1}{3} & yz \\ xz & yz & z^2-\frac{1}{3} \end{pmatrix}.$$

The point $A(x, y, z)$ lies on S since,

$$traceA_{(x,y,z)} = \sqrt{\frac{3}{2}}(x^2+y^2+z^2-1) = 0$$

and $traceA^2{}_{(x,y,z)} = \|f(x,y,z)\|^2$

$$= \frac{3}{2}\left(x^4+y^4+z^4-\frac{2}{3}(x^2+y^2+z^2)+\frac{1}{3}\right)+3(x^2y^2+x^2z^2+y^2z^2)$$
$$= \frac{3}{2}(x^2+y^2+z^2)^2-(x^2+y^2+z^2)+\frac{1}{2}$$
$$= \frac{3}{2}-1+\frac{1}{2} = 1.$$

The eigenvalues of $A(x, y, z)$ are the solutions of the system.

$$traceA_{(x,y,z)} = 0, traceA^2{}_{(x,y,z)} = 1, traceA^3{}_{(x,y,z)} = \frac{1}{\sqrt{6}}$$

That is, if the eigenvalues of $A(x, y, z)$ are $\lambda_1\ \lambda_2, \lambda_3$, then

$$\lambda_1 + \lambda_2 + \lambda_3 = 0$$
$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$$
$$\lambda_1^3 + \lambda_2^3 + \lambda_3^3 = \frac{1}{\sqrt{6}}.$$

Since the embedding f is 2- dimensional, the matrix $A(x, y, z)$ only has two distinct eigenvalues, say $\lambda_2 = \lambda_3$. Using such a fact simplifies the problem of finding the eigenvalues which are $\lambda_1 = \frac{2}{\sqrt{6}}, \lambda_2 = -\frac{1}{\sqrt{6}}, \lambda_3 = -\frac{1}{\sqrt{6}}$.

Conversely, let,

$$D = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}} \end{pmatrix} = A_{(1,0,0)}$$

Also let,

$$P = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \in O(3).$$

So P is orthogonal, i.e. $PP^T = I_3$. Then

$$PDP^T = \sqrt{\frac{3}{2}} \begin{pmatrix} u_1^2 - \frac{1}{3} & u_1 v_1 & u_1 w_1 \\ u_1 v_1 & v_1^2 - \frac{1}{3} & v_1 w_1 \\ u_1 w_1 & v_1 w_1 & w_1^2 - \frac{1}{3} \end{pmatrix} = A_{(u_1, v_1, w_1)}$$

So $f(\mathrm{P}^2)$ can be identified by the set

$$\ell = \{PDP^T : P \in O(3)\}.$$

Also if $R \in O(3)$ such that $RDR^T = D$ then

$$R = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & v_2 & v_3 \\ 0 & w_2 & w_3 \end{pmatrix}, where \begin{pmatrix} v_2 & v_3 \\ w_2 & w_3 \end{pmatrix} \in O(2)$$

That is, P^2 is identified with $O(3)/O(2) \times O(1)$

Claim 1: The matrix $QR\Delta R^T Q^T$ is normal to ℓ at QDQ^T where Δ is diagonal and $RDR^T = D$ For, let $s \longmapsto P(s)\, DP^T(s)$ be a path in ℓ through D such that $P(0) = \mathrm{Q}$.

A tangent to ℓ at QDQ^T is

$$(\dot{P}DP^T + PD\dot{P}^T)\,|_{s=0} = \dot{P}DQ^T + QD\dot{P}^T.$$

Now $PP^T = I_3$ and so

$$\dot{P}P^T + P\dot{P}^T = 0$$

Hence at $s = 0, \dot{P}^T = -Q^T \dot{P} Q^T$. Thus, a tangent to ℓ at QDQ^T is

$$\dot{P}DQ^T - QDQ^T \dot{P} Q^T$$

Now

$$\begin{aligned} &< \dot{P}DQ^T - QDQ^T \dot{P}Q^T, QR\Delta R^T Q^T > \\ &= trace\dot{P}DQ^T QR\Delta R^T Q^T - traceQDQ^T \dot{P} Q^T QR\Delta R^T Q^T \\ &= trace\dot{P}DR\Delta R^T Q^T - traceQDQ^T \dot{P}R\Delta R^T Q^T \\ &= trace\dot{P}DR\Delta R^T Q^T - trace\dot{P}R\Delta R^T Q^T QDQ^T \\ &= trace\dot{P}DR\Delta R^T Q^T - trace\dot{P}R\Delta DR^T Q^T \\ &= trace\dot{P}DR\Delta R^T Q^T - trace\dot{P}RD\Delta R^T Q^T \\ &= trace\dot{P}DR\Delta R^T Q^T - trace\dot{P}DR\Delta R^T Q^T = 0 \end{aligned}$$

In particular, $R\Delta R^T$ is normal to ℓ at D. Thus, the equation $R\Delta R^T = PDP^T$ has a solution corresponding to $\Delta = D$ and $P = R$ where R as above.There are infinite choices of R, which implies that the intersection between $f(\mathrm{P}^2)$ and the affine normal plane of $f(\mathrm{P}^2)$ at $f(1,0,0)$ is infinite. Since f is an embedding, f is not transnormal (a fact which is already known since $\chi(P^2) = 1$).

The aim now is to find four orthonormal vectors normal to f at $f(x, y, z)$. Starting at the point $f(1, 0, 0)$, one unit normal corresponds to $RDR^T = D$ itself, which is $v_1(1,0,0) = f(1,0,0)$.

To find the other three in the orthonormal set, assume that Δ_1 is diagonal and $R\Delta_1 R^T \perp D$. Then $trace\Delta_1 D = trace\Delta_1 R^T DR = traceR\Delta_1 R^T D = 0$.

Thus, if $\Delta_1 = diagonal\ (a, b, c)$ then $2a - b - c = 0$.

Let

$$R = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}.$$

Then

$$R\Delta_1 R^T = \begin{pmatrix} a & 0 & 0 \\ 0 & b\cos^2\theta + c\sin^2\theta & (c-b)\sin\theta\cos\theta \\ 0 & (c-b)\sin\theta\cos\theta & b\sin^2\theta + c\cos^2\theta \end{pmatrix}$$

Let $b = 2s$, $c = 2t$ then $a = s + t$.

Thus,

$$R\Delta_1 R^T = (s+t)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (t-s)\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2\cos 2\theta & \sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{pmatrix},$$

or

$$R\Delta_1 R^T = (s+t)I_3 + (t-s)A\cos 2\theta + (t-s)B\sin 2\theta.$$

where,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Thus, the normal plane of f at $f(1, 0, 0)$ is spanned by the normals corresponding to the matrices D, I_3, A, B. To generalize the situation at any point on f, assume that

$$Q = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \in O(3).$$

Then the corresponding matrices for the required normals are QDQ^T, I_3, QAQ^T, QBQ^T

The first matrix QDQ^T- is the point itself and so the first unit normal is $v_1 = f$

The second unit normal corresponds to I_3 and so,

$$v_2 = \frac{1}{\sqrt{3}}(1,1,1,0,0,0)$$

The third and fourth normals correspond to the matrices.

$$QAQ^T = \begin{pmatrix} u_3{}^2 - u_2{}^2 & u_3v_3 - u_2v_2 & u_3w_3 - u_2w_2 \\ u_3v_3 - u_2v_2 & v_3{}^2 - v_2{}^2 & v_3w_3 - v_2w_2 \\ u_3w_3 - u_2w_2 & v_3w_3 - v_2w_2 & w_3{}^2 - w_2{}^2 \end{pmatrix}$$

and

$$QBQ^T = \begin{pmatrix} 2u_2u_3 & u_2v_3 + u_3v_2 & u_2w_3 + u_3w_2 \\ u_2v_3 + u_3v_2 & 2v_2v_3 & v_2w_3 + v_3w_2 \\ u_2w_3 + u_3w_2 & v_2w_3 + v_3w_2 & 2w_2w_3 \end{pmatrix}.$$

One well known orthogonal 3 × 3 matrix is;

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ \sin\theta\cos\psi & -\sin\psi & \cos\theta\cos\psi \\ \sin\theta\sin\psi & \cos\psi & \cos\theta\sin\psi \end{pmatrix}$$

In terms of x, y, z the above orthogonal matrix can be rewritten as;

$$\begin{pmatrix} x & 0 & -\sqrt{y^2+z^2} \\ y & -\dfrac{z}{\sqrt{y^2+z^2}} & \dfrac{xy}{\sqrt{y^2+z^2}} \\ z & \dfrac{y}{\sqrt{y^2+z^2}} & \dfrac{xz}{\sqrt{y^2+z^2}} \end{pmatrix}$$

Thus,

$$QAQ^T = \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & \dfrac{x^2y^2 - z^2}{y^2+z^2} & \dfrac{x^2yz + yz}{y^2+z^2} \\ -xz & \dfrac{x^2yz + yz}{y^2+z^2} & \dfrac{x^2z^2 - y^2}{y^2+z^2} \end{pmatrix}$$

and

$$QBQ^T = \begin{pmatrix} 0 & z & -y \\ z & \dfrac{-2xyz}{y^2+z^2} & \dfrac{xy^2 - xz^2}{y^2+z^2} \\ -y & \dfrac{xy^2 - xz^2}{y^2+z^2} & \dfrac{2xyz}{y^2+z^2} \end{pmatrix}.$$

Hence the other two unit normals are;

$$v_3 = \frac{1}{\sqrt{2}}\left(y^2+z^2, \frac{x^2y^2 - z^2}{y^2+z^2}, \frac{x^2z^2 - y^2}{y^2+z^2}, -\sqrt{2}xy, -\sqrt{2}xz, \frac{\sqrt{2}(x^2yz + yz)}{y^2+z^2}\right)$$

and

$$v_4 = \frac{1}{\sqrt{2}}\left(0, \frac{-2xyz}{y^2+z^2}, \frac{2xyz}{y^2+z^2}, \sqrt{2}z, -\sqrt{2}y, \frac{\sqrt{2}(xy^2 - xz^2)}{y^2+z^2}\right)$$

Now consider the partial tube in $\mathbb{R}^6$ identified by;

$$\rho = \{PDP^T + \xi\cos\psi PAP^T + \xi\sin\psi PBP^T : P \in O(3)\},$$

where $\xi > 0$ and $\psi \in [0, 2\pi]$. Thus, the partial tube is based at f and is built by circles of radii ξ in the normal plane spanned by v_3 and v_4 at every point on f.

Now let,

$$\bar{D} = D + \xi\cos\psi A + \xi\sin\psi B.$$

Then,

$$\bar{D} = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} - \xi\cos\psi & \xi\sin\psi \\ 0 & \xi\sin\psi & -\frac{1}{\sqrt{6}} + \xi\cos\psi \end{pmatrix}$$

The eigenvalues of $\bar{D}$ are,

$$\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}} - \xi, \frac{-1}{\sqrt{6}} + \xi.$$

Since the partial tube is a 3-dimensional manifold, the eigenvalues should be distinct, hence $\xi \neq \frac{3}{\sqrt{6}}$. Such a condition can be easily satisfied since ξ needs to be small to ensure that the partial tube is embedded. Also $\bar{D} = EJE^T$

where

$$J = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{-1}{\sqrt{6}} - \xi & 0 \\ 0 & 0 & \frac{-1}{\sqrt{6}} + \xi \end{pmatrix} \text{ and } E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\psi}{2} & \sin\frac{\psi}{2} \\ 0 & -\sin\frac{\psi}{2} & \cos\frac{\psi}{2} \end{pmatrix}$$

Let $H=PE$, so H is orthogonal. Then the partial tube is identified with,

$$P\bar{D}P^T = PEJE^T P^T = HJH^T.$$

Also if $R \in O(3)$ such that $RDR^T = D$ and $RJR^T =$ J, then

$$R(D + \xi A)R^T = D + \xi A$$

or

$$RDR^T + \xi RAR^T = D + \xi A.$$

Thus, $RAR^T = A$ and so

$$R = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$$

That is, the domain of ρ is identified with $O(3)/O(1) \times O(1) \times O(1)$.

Claim 2: The matrix $Q\Delta Q^T$ is normal to ρ at QJQ^T where Δ is diagonal.

For, let $s \mapsto H(s)\ J\ H^T(s)$ be a path in ρ through J such that $H(0)$=Q.

A tangent to ρ at QJQ^T is,

$$(\dot{H}JH^T + HJ\dot{H}^T)|_{s=0} = \dot{H}JQ^T + QJ\dot{H}^T$$

Now $HH^T = I_3$ and so $\dot{H}H^T + H\dot{H}^T = 0$.

Hence at $s = 0$, $\dot{H}^T = -Q^T\dot{H}Q^T$ Thus, a tangent to ℓ at QJQ^T is,

$$\dot{H}JQ^T - QJQ^T\dot{H}Q^T.$$

Now,

$$\begin{aligned} &< \dot{H}JQ^T - QJQ^T\dot{H}Q^T, Q\Delta Q^T > \\ &= trace\dot{H}JQ^T Q\Delta Q^T - traceQJQ^T\dot{H}Q^T Q\Delta Q^T \\ &= trace\dot{H}J\Delta Q^T - traceQJQ^T\dot{H}\Delta Q^T \\ &= trace\dot{H}J\Delta Q^T - trace\dot{H}\Delta Q^T QJQ^T \\ &= trace\dot{H}J\Delta Q^T - trace\dot{H}\Delta JQ^T \\ &= trace\dot{H}J\Delta Q^T - trace\dot{H}J\Delta Q^T = 0 \end{aligned}$$

Assume that Δ_1 is diagonal such that $\Delta_1 \perp D$ and $\Delta_1 \perp J$ Thus,

$$trace\Delta_1 J = \frac{2a - b - c}{\sqrt{6}} + (c - b)\xi = 0$$

But $2a - b - c = 0$. Hence $b =$ and so Δ_1 - bI_3.

Consider the equation

$HJH^T = Q\Delta Q^T$

If $\Delta = \Delta_1$, then

$HJH^T = Q\Delta_1 Q^T = bI_3$

and so $J = bI_3$, which is false. Now consider the six matrices obtained from J by the different permutations of the eigenvalues of J, say J_1,......J_6. Let R_i by the matrix obtained by changing the rows of R such that $R_i J R_i^T$, $i=1, \ldots, 6$.

Now $\Delta = J_i$, $i=1, \ldots, 6$. Thus,

$HJH^T = Q\,J_i Q^T$

or

$Q^T HJ(Q^T H)^T = J_i$.

Hence

$H = QR_i$

Thus, we have six solutions, say H_k, $k = 1, \ldots, 6$ with the same normal plane at each, namely QJQ^T. Thus, the above partial tube in $\mathbb{R}^6$ is a 6-transnormal embedding. Also the points in the generating frame lie on a circle.

Upon the process of generalization of this example, $\mathbb{R}^6$ can be replaced by $\mathbb{R}^m$ where,

$m = \dfrac{n(n+1)}{2}$, $n \geq 4$. The next suggested example will be the embedding of P^2 in $\mathbb{R}^{10}$. There the treasure will be about: a tale of 8 normals.

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