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Fitting Second-Order Model to Mixed Three-Level and Four-Level Factorial Designs Using the Coefficient of Polynomial Contrast

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Abstract: This paper deals with mixed three-level and four-level factorial designs. A proposed procedure uses the coefficient of polynomial contrast for fitting response surface models to mixed three-level and four-level factorial designs. This procedure avoids using the least squares method, which is particularly cumbersome when the number of independent variables is more than two, The new formulae are fixed for all mixed three-level and four-level factorial designs regardless of the number of factors. and also allows the possibility of calculating each coefficient of the model individually. The results of the proposed procedure for linear, quadratic, interaction are in agreement with the results of least squares method. The procedure provides a quick and easy way to fit response surface models to mixed three-level and four-level factorial designs and to verify the result of any coefficient without disturbing other coefficients in the model. Statistical packages are available for analyzing response surface models. However, these packages are either expensive or require a specific skill set to perform the analysis. This paper could motivate researchers to study the possibility of applying a fixed formula to all factorial designs.

Key words: Three-level factorial design · Four-level factorial design · Response surface methodology

INTRODUCTION

Response surface models are widely used to describe the behavior of different experiments in different fields. In the twenties of last century Fisher announced the birth of a new subject called factorial experiment [1]. A significant contribution was given by Yates in 1937 for analyzing two-level designs [2, 3]. Then Davies developed this procedure to fit a second-order response surface model to three-level designs [3]. The interest moved to analyzing and fitting response surface models to mixed design. For instance, Margolin [4] developed the procedure given by Yates (two-level factorial design) and Davies (three-level factorial design) for analyzing and fitting response surface model to mixed two-level and three-level factorial designs. Fitting mixed two-level and three-level factorial designs and mixed two-level and four-level factorial designs were studied by Draper & Stoneman [5] discussing the number of runs needed to fit response surface models to these types of designs.

The contribution to factorial experiments increased day after day. This encourages the researcher to review the contributions. The contribution that had been published from 1957 was reviewed by Herzberg & Cox [6] to provide detailed bibliography regarding factorial designs. Addelman [1] reviewed the contribution from 1957 to 1972 and found that most of the contributions were given to fractional factorial designs. Edmondson [7] suggested a new procedure by using pseudo-factors to represent four-level for fitting a second-order model to four-level design. Bisgaard [8] presented a method for accommodating four-level factors in two-level designs. This method converts two or more columns to accommodate multi-level factors. Analyzing two-level design was studied by Abbas' *et al.* [9] suggesting a new procedure for analyzing this type of experiment. Furthermore, a new procedure for analyzing and fitting response surface model to three-level factorial designs was given by Abbas [10]. Analyzing and fitting response surface to mixed two-level and three-level factorial

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designs was studied by Abbas & Low [11]. Analyzing four-level design was studied by Wasin *et al.* [12] suggesting new formulae for fitting this type of experiment by using the coefficients of orthogonal contrast.

Fitting response surface models to mixed three-level and four-level factorial designs and to provide fixed formulae for linear, quadratic and interactions coefficients is the objective of this research in order to avoid the complicated procedure in using the least squares method and to estimate each coefficient independently as well. Thus, the main interest of this research is to provide a quick and easy way for fitting response surface models to mixed three-level and four-level factorial designs.

Mixed Three-Level and Four-Level Factorial Designs: Three-level and four-level factorial experiment is a factorial design with mixed levels, m factors each at three levels and q factors each at four levels denoted by 3^m4^q. The simplest design for three-level and four-level factorial design is the design with two factors one at three levels and second at four levels. The total number of runs required for 3^14^1 is 12 runs for one replicate [3, 13].

Proposed Procedure: Fitting response surface models to 3^m4^q experiment using the proposed procedure can be achieved by using the coefficients of orthogonal contrast (-3 -1 1 3) to find the linear coefficients and the interaction between different factors and (1 -1 -1 1) to find the quadratic coefficients for four-level factor and coefficients of orthogonal contrast for three-level factor is (-1 0 1) to find the linear coefficients for three-level factor and the interaction between three-level factors and (1 -2 1) to find the quadratic coefficients. The coefficients for the interaction between three-level and four-level factors can be found by using the orthogonal contrast for three and four levels.

The proposed procedure for fitting a response surface model to mixed three-level and four-level factorial design depends on combining the two procedures for fitting experiments of type 3^m [9] and experiments of type 4^q [12].

The formulae for three-level factors are

Linear coefficient

$$b_l = \frac{Linear\ \ contrast\ \ for\ A_l}{2n}, \quad l = 1,\ 2,\ ...,\ m$$

Quadratic coefficient

$$b_{ll} = \frac{Quadratic\ \ contrast\ \ for\ A_l}{2n},$$

Interaction coefficient

$$b_{lL} = \frac{Linear\ \ contrast\ \ for\ \ A_l A_L}{4 \times n}, \quad l \neq L$$

and the formulae for four-level designs are

Linear coefficient

$$\gamma_L = \frac{Linear\ \ contrast\ \ for\ A_L}{20 \times n},$$

Quadratic coefficient

$$\gamma_{LL} = \frac{Quadratic\ \ contrast\ \ for\ A_L}{16 \times n},$$

Interaction coefficient

$$\beta_{LQ} = \frac{Linear\ \ contrast\ \ for\ A_L A_Q}{400 \times n}, \quad L \neq Q$$

The formula for calculating the coefficients of the interaction between factors that have three levels and factors that have four levels is given in equation (1).

$$\beta_{lL} = \frac{Linear\ contrast\ \ for\ A_l A_L}{40 \times n}, \tag{1}$$

where β_{lL} represents the regression coefficient for factor l which has three levels and factor L which has four levels and n represents the number of replicates at the joint levels.

To find the interaction between different factors C^m_1 C^q_1 experiments of the form 3^14^1 should be studied to find the interaction between factors that have three levels and factors that have four levels.

The formula for the intercept b_o for mixed three-level and four-level factorial design is given in equation (2).

$$b_0 = \bar{Y} - b_{11}\bar{X}_1 - ... - b_{mm}\bar{X}_m - \gamma_{11}\bar{Z}_1 - ... - \gamma_{qq}\bar{Z}_q \tag{2}$$

where $\bar{Z} = \frac{\sum Z_i^2}{k}, \bar{X} = \frac{\sum X_i^2}{k}$ and k is the total number of observations.

Proposed Formula: Suppose there are m factors each at three levels (X_1, X_2, ..., X_m) and q factors each at four levels (Z_1, Z_2, ..., Z_q). Consider a second-order response surface model as given below:

$$
\begin{aligned}
Y_i = b_0 + b_1 X_{1i} + b_2 X_{2i} + \ldots + b_m X_{mi} + b_{12} X_{1i} X_{2i} + \ \ldots + b_{(m-1)m} X_{(m-1)i} X_m \\
b_{11} X_{1i}^2 + \ \ldots + b_{mm} X_{mi}^2 + \gamma_1 Z_{1i} + \gamma_2 Z_{2i} + \ \ldots + \gamma_q Z_{qi} + \gamma_{11} Z_{1i}^2 + \ \ldots + \gamma_{qq} Z_{qi}^2 \\
+\gamma_{12} Z_{1i} Z_{2i} + \ \ldots + \gamma_{(q-1)q} Z_{(q-1)i} Z_{qi} + \beta_{11} X_{1i} Z_{1i} + \ \ldots + \beta_{mq} X_{mi} Z_{qi}
\end{aligned}
$$

The model obeys some constraints.

- Constraints for the factors at three levels

1- $\sum_{i=1}^{k} X_i = 0$ 2- $\sum_{i<j}^{k} X_i X_j = 0$ 3- $\sum_{i<j}^{k} X_i^2 X_j = 0$ 4- $\sum_{i<j}^{k} X_i X_j^2 = 0$

5- $\sum_{i<j<h}^{k} X_i X_j X_h = 0$ 6- $\sum_{i=1}^{k} X_i^2 = \sum_{i=1}^{k} X_i^4 = 2 \times 3^{m-1}$ 7- $\sum_{i<j}^{k} (X_i X_j)^2 = 4 \times 3^{m-2}$

- Constraints for the factors at four levels

1- $\sum_{i=1}^{k} Z_i = 0$ 2- $\sum_{i<j}^{k} Z_i Z_j = 0$ 3- $\sum_{i=1}^{k} Z_i Z_i^2 = 0$ 4- $\sum_{i<j}^{k} Z_i Z_j^2 = 0$

5- $\sum_{i=1}^{k} Z_i^2 = 20 \times 4^{q-1}$ 6- $\sum_{i=1}^{k} Z_i^4 = 164 \times 4^{q-1}$ 7- $\sum_{i<j}^{k} Z_i^2 Z_j^2 = 400 \times 4^{q-2}$ 8- $\sum_{i<j<z}^{k} Z_i Z_j Z_z = 0$ 9- $\sum_{i<j}^{k} (Z_i Z_j)^2 = 400 \times 4^{q-2}$

- Constraint for the joint effects between factors at three levels and factors at four levels

1- $\sum_{i=1}^{k} (Z_i X_i)^2 = 40 \times (4^{q-1} \times 3^{m-1})$ 2- $\sum_{i=1}^{k} X_i Z_i = \sum_{i=1}^{k} X_i^2 Z_i = \sum_{i=1}^{k} X_i Z_i^2 = 0$ 3- $\sum_{i<j<l}^{k} X_i X_j Z_l = \sum_{i<j<l}^{k} Z_i Z_j X_l = 0$

The constraints are obtained from the coefficients of orthogonal contrasts.

To illustrate the procedure, consider a $3^2 4^2$ experiment without losing information for the general case.

Suppose there are four factors: X_1, X_2 at three levels and Z_1, Z_2 at four levels. The model for this experiment is given in equation (3).

$$
\begin{aligned}
Y_i = b_0 + b_1 X_{1i} + b_2 X_{2i} + b_{11} X_{1i}^2 + b_{22} X_{2i}^2 + b_{12} X_{1i} X_{2i} + \gamma_1 Z_{1i} + \gamma_2 Z_{2i} + \gamma_{11} Z_{1i}^2 + \gamma_{22} Z_{2i}^2 \\
+\gamma_{12} Z_{1i} Z_{2i} + \beta_{11} X_{1i} Z_{1i} + \beta_{12} X_{1i} Z_{2i} + \beta_{21} X_{2i} Z_{1i} + \beta_{22} X_{2i} Z_{2i} \\
i = 1, \ 2, \ \ldots, \ k
\end{aligned}
\tag{3}
$$

The treatment combinations for this experiment are given in Table 1.

The levels of each factor represent the coded form which is the linear coefficients of the orthogonal contrast. The relationship between actual and coded variables for three levels is:

$$X = \frac{C - (High + Low)/2}{(High - Low)/2}$$

and for four levels is

$$X = \frac{C - \text{Average of all levels } (a_1 + a_2 + a_3 + a_4)/4}{(\text{Range of any two consecutive levels})/2}$$

Table 1: A design with four factors where X_1 and X_2 have three levels each and Z_1 and Z_2 have four levels each

		X_1	-1	-1	-1	0	0	0	1	1	1
		X_2	-1	0	1	-1	0	1	-1	0	1
Z_1	Z_2										
-3	-3		Y_1	Y_{17}	Y_{33}	Y_{49}	Y_{65}	Y_{81}	Y_{97}	Y_{113}	Y_{129}
-1	-3		Y_2	Y_{18}	Y_{34}	Y_{50}	Y_{66}	Y_{82}	Y_{98}	Y_{114}	Y_{130}
1	-3		Y_3	Y_{19}	Y_{35}	Y_{51}	Y_{67}	Y_{83}	Y_{99}	Y_{115}	Y_{131}
3	-3		Y_4	Y_{20}	Y_{36}	Y_{52}	Y_{68}	Y_{84}	Y_{100}	Y_{116}	Y_{132}
-3	-1		Y_5	Y_{21}	Y_{37}	Y_{53}	Y_{69}	Y_{85}	Y_{101}	Y_{117}	Y_{133}
-1	-1		Y_6	Y_{22}	Y_{38}	Y_{54}	Y_{70}	Y_{86}	Y_{102}	Y_{118}	Y_{134}
1	-1		Y_7	Y_{23}	Y_{39}	Y_{55}	Y_{71}	Y_{87}	Y_{103}	Y_{119}	Y_{135}
3	-1		Y_8	Y_{24}	Y_{40}	Y_{56}	Y_{72}	Y_{88}	Y_{104}	Y_{120}	Y_{136}
-3	1		Y_9	Y_{25}	Y_{41}	Y_{57}	Y_{73}	Y_{89}	Y_{105}	Y_{121}	Y_{137}
-1	1		Y_{10}	Y_{26}	Y_{42}	Y_{58}	Y_{74}	Y_{90}	Y_{106}	Y_{122}	Y_{138}
1	1		Y_{11}	Y_{27}	Y_{43}	Y_{59}	Y_{75}	Y_{91}	Y_{107}	Y_{123}	Y_{139}
3	1		Y_{12}	Y_{28}	Y_{44}	Y_{60}	Y_{76}	Y_{92}	Y_{108}	Y_{124}	Y_{140}
-3	3		Y_{13}	Y_{29}	Y_{45}	Y_{61}	Y_{77}	Y_{93}	Y_{109}	Y_{125}	Y_{141}
-1	3		Y_{14}	Y_{30}	Y_{46}	Y_{62}	Y_{78}	Y_{94}	Y_{110}	Y_{126}	Y_{142}
1	3		Y_{15}	Y_{31}	Y_{47}	Y_{63}	Y_{79}	Y_{95}	Y_{111}	Y_{127}	Y_{143}
3	3		Y_{16}	Y_{32}	Y_{48}	Y_{64}	Y_{80}	Y_{96}	Y_{112}	Y_{128}	Y_{144}

Based on the proposed procedure, there are two types of factors, three-level factors and four-level factors. So three-level factors can be considered as 3^2 factorial design and four-level factors can be considered as 4^2 factorial design. Interaction between factors that have three levels and factors that have four levels can be studied by constructing $C_1^m\ C_1^q$ experiments of the form $3^1 4^1$. The procedure is illustrated below:

First, separate the experiment into two experiments 3^2 and 4^2 and analyze each experiment using the formulae given earlier.

Second, in order to complete the analysis, interaction coefficient between factors at three levels and factors at four levels should be obtained and the intercept formula as well.

The formula for b_0 in equation (3) can be derived by summing equation (3) over i and applying the constraints will result in the formula for b_0.

$$b_0 = \bar{Y} - b_{11}\bar{X}_1 - b_{22}\bar{X}_2 - \gamma_{11}\bar{Z}_1 - \gamma_{22}\bar{Z}_2$$

where , $\bar{Z} = \frac{\sum Z_i^2}{k}, \bar{X} = \frac{\sum X_i^2}{k}$ and k is the total number of observations.

The formula for the interaction coefficient between factors at three levels and four levels can be derived by multiplying equation (3) by X_1Z_1 and summing over i, which will give:

$$\sum X_{1i}Z_{1i}Y_i = \beta_{11}\sum (X_{1i}Z_{1i})^2$$

i.e.,

$$\beta_{11} = \frac{\sum X_{1i}Z_{1i}Y_i}{\sum (X_{1i}Z_{1i})^2}$$

Similarly for β_{12}, β_{21} and β_{22}.

In general, the formula is given below:

$$\beta_{lL} = \frac{\sum X_{li}Z_{Li}Y_i}{\sum (X_{li}Z_{Li})^2}, \quad l = 1,\ 2,\ ...,m,\ \ L = 1,\ 2,\ ...,q$$

This formula can be written in the form of contrast. Consider the formula for β_{11}:

$$\beta_{11} = \frac{\sum X_{1i}Z_{1i}Y_i}{\sum (X_{1i}Z_{1i})^2} \tag{4}$$

The denominator of equation (4) is equal to

$\Sigma(X_{1i}Z_{1i})^2 = 40\ (4^{q-1}3^{m-1}) = 40\ (4^1 3^1) = 480$. This is equal to 40×12, where 12 represents the number of replicates at each joint level.

The numerator of equation (4) is:

$$\sum_{i=1}^{144} X_{1i}Z_{1i}Y_i = (-1)(-3)Y_1 + (-1)(-1)Y_2 + (-1)(1)Y_3 + (-1)(3)Y_4 + \ ... + (-1)(3)Y_{48}$$
$$+(0)(-3)Y_{49} + \ ... + (0)(3)Y_{96} + (1)(-3)Y_{97} + \ ... + (1)(3)Y_{164}$$

$$= 3Y_1 + Y_2 - Y_3 - 3Y_4 + \ ... - 3Y_{48} - Y_{97} - Y_{98} + Y_{99} +$$
$$3Y_{100} + \ ... - 3Y_{141} - Y_{142} + Y_{143} + 3Y_{144}$$

$$\begin{aligned} =3(&Y_1 + Y_5 + Y_9 + Y_{13} + Y_{17} + Y_{21} + Y_{25} + Y_{29} + Y_{33} + Y_{37} + Y_{41} + Y_{45} + Y_{100} + Y_{104} \\ &+Y_{108} + Y_{112} + Y_{116} + Y_{120} + Y_{124} + Y_{128} + Y_{132} + Y_{136} + Y_{140} + Y_{144}) \\ &+(Y_2 + Y_6 + Y_{10} + Y_{14} + Y_{18} + Y_{22} + Y_{26} + Y_{30} + Y_{34} + Y_{38} + Y_{42} + Y_{46} + Y_{99} \\ &+Y_{103} + Y_{107} + Y_{111} + Y_{115} + Y_{119} + Y_{123} + Y_{127} + Y_{131} + Y_{135} + Y_{139} + Y_{143}) \\ &-(Y_3 + Y_7 + Y_{11} + Y_{15} + Y_{19} + Y_{23} + Y_{27} + Y_{31} + Y_{35} + Y_{39} + Y_{43} + Y_{47} + Y_{98} \\ &+Y_{102} + Y_{106} + Y_{110} + Y_{114} + Y_{118} + Y_{122} + Y_{126} + Y_{130} + Y_{134} + Y_{138} + Y_{142}) \\ &-3(Y_4 + Y_8 + Y_{12} + Y_{16} + Y_{20} + Y_{24} + Y_{28} + Y_{32} + Y_{36} + Y_{38} + Y_{44} + Y_{48} + Y_{97} \\ &+Y_{101} + Y_{105} + Y_{109} + Y_{113} + Y_{117} + Y_{121} + Y_{125} + Y_{129} + Y_{133} + Y_{137} + Y_{141}) \end{aligned} \tag{5}$$

The same result can be obtained if the contrast is used. In order to show that the numerator of equation (4) is equal to the joint linear contrast between X_1 and Z_1, experiment of type $3^1 \times 4^1$ where one factor has three levels and the other has four levels will be considered.

Table 2: The results for factors X_1 (at three levels) and Z_1 (at four levels)

Z_1	X_1	Response
-3	-1	$Y_1+Y_5+Y_9+Y_{13}+Y_{17}+Y_{21}+Y_{25}+Y_{29}+Y_{33}+Y_{37}+Y_{41}+Y_{45}$
-1	-1	$Y_2+Y_6+Y_{10}+Y_{14}+Y_{18}+Y_{22}+Y_{26}+Y_{30}+Y_{34}+Y_{38}+Y_{42}+Y_{46}$
1	-1	$Y_3+Y_7+Y_{11}+Y_{15}+Y_{19}+Y_{23}+Y_{27}+Y_{31}+Y_{35}+Y_{39}+Y_{43}+Y_{47}$
3	-1	$Y_4+Y_8+Y_{12}+Y_{16}+Y_{20}+Y_{24}+Y_{28}+Y_{32}+Y_{36}+Y_{40}+Y_{44}+Y_{48}$
-3	0	$Y_{49}+Y_{53}+Y_{57}+Y_{61}+ Y_{65}+Y_{69}+Y_{73}+Y_{77}+Y_{81}+Y_{85}+Y_{89}+Y_{93}$
-1	0	$Y_{50}+Y_{54}+Y_{58}+Y_{62}+Y_{66}+Y_{70}+Y_{74}+Y_{78}+Y_{82}+Y_{86}+Y_{90}+Y_{94}$
1	0	$Y_{51}+Y_{55}+Y_{59}+Y_{63}+Y_{67}+Y_{71}+Y_{75}+Y_{79}+Y_{83}+Y_{87}+Y_{91}+Y_{95}$
3	0	$Y_{52}+Y_{56}+Y_{60}+Y_{64}+Y_{68}+Y_{72}+Y_{76}+Y_{80}+Y_{84}+Y_{88}+Y_{92}+Y_{96}$
-3	1	$Y_{97}+Y_{101}+Y_{105}+Y_{109}+Y_{113}+Y_{117}+Y_{121}+Y_{125}+Y_{129}+Y_{133}+Y_{137}+Y_{141}$
-1	1	$Y_{98}+Y_{102}+Y_{106}+Y_{110}+Y_{114}+Y_{118}+Y_{122}+Y_{126}+Y_{130}+Y_{134}+Y_{138}+Y_{142}$
1	1	$Y_{99}+Y_{103}+Y_{107}+Y_{111}+Y_{115}+Y_{119}+Y_{123}+Y_{127}+Y_{131}+Y_{135}+Y_{139}+Y_{143}$
3	1	$Y_{100}+Y_{104}+Y_{108}+Y_{112}+Y_{116}+Y_{120}+Y_{124}+Y_{128}+Y_{132}+Y_{136}+Y_{140}+Y_{144}$

Summing the observations in the cells, which have the same joint level before finding the joint contrast is shown in Table 2 for X_1Z_1. Similar tables can be constructed for X_1Z_2, X_2Z_1 and X_2Z_2.

From Table 2, the linear joint contrast for X_1Z_1 is:

$$=(-3)\ (-1)\ [Y_1+Y_5+Y_9+Y_{13}+Y_{17}+Y_{21}+Y_{25}+Y_{29}+Y_{33}+Y_{37}+Y_{41}+Y_{45}]$$
$$+(-1)\ (-1)\ [Y_2+Y_6+Y_{10}+Y_{14}+Y_{18}+Y_{22}+Y_{26}+Y_{30}+Y_{34}+Y_{38}+Y_{42}+Y_{48}]+$$
$$\ldots+(3)\ (1)\ [Y_{100}+Y_{104}+Y_{108}+Y_{112}+Y_{116}+Y_{120}+Y_{124}+Y_{128}+Y_{132}+Y_{136}+$$
$$Y_{140}+Y_{144}]$$

Comparing this result with equation (5), revealed that the two results are the same which means that the numerator of equation (4) can be written as a linear joint contrast for X_1 and Z_1. Thus equation (4) can be written by using the linear contrast as given below:

$$\beta_{11}=\frac{\sum X_{1i}Z_{1i}Y_i}{\sum (X_{1i}Z_{1i})^2}=\frac{Linear\ \ contrast\ \ for\ \ X_1Z_1}{40\times 12}$$

where 12 represents the number of replicates at each joint level and A_1 represents the factor at two levels and B_1 represents the factor at four levels. the formula for other coefficients.

In general, let n represents the number of replicates at the joint levels. Then the formula becomes:

$$\beta_{lL}=\frac{Linear\ contrast\ \ for\ A_lA_L}{40\times n} \qquad l\neq L$$

Application: This application is only to illustrate the implementation of the new formulae. The effect of three factors, namely *pH* (X_1) at four levels (2, 4, 6 and 8), Cation concentration (X_2) at four levels (0.4, 0.6, 0.8 and 1 mM) and pectin dosage (X_3) at three levels (1, 4.5 and 8 mg/L) on flocculating activity as a response was studied.

Table 3: The actual and coded form for the selected variables

pH	2	4	6	8
Cation concentration	0.4	0.6	0.8	1
Coded	-3	-1	1	3
Pectin dosage	1	4.5	8	
Coded	-1	0	1	

The design used to run this experiment is 3^14^2 design. The total number of runs is 48-run. The levels of each factor in actual and coded form are given in Table 3.

The researcher wants to fit a second-order response surface model to this experiment. The model is given in equation (6).

$$Y=b_0+b_1X_1+b_2X_2+b_3X_3+b_{11}X_1^2+b_{22}X_2^2+ b_{33}X_3^2+b_{12}X_1X_2+b_{13}X_1X_3+b_{23}X_2X_3 \tag{6}$$

Two methods will be used to fit the model in equation (6). The methods are the least squares method and the proposed procedure for fitting this type of experiments.

Proposed Procedure: Fitting a response surface model to this type of experiment using proposed procedure requires separating this experiment into two experiments, one experiment of type 3^1 and another experiment of type 4^2.

Based on the proposed procedure, either experiment can be analyzed first. So 4^2 experiment will be analyzed first. Four-level experiment can be analyzed using the formula given by Wasin *et al.* [12]. This requires dealing with each factor as an experiment of type 4^1 to find the linear and quadratic coefficients and then study 4^2 experiment to find the interaction between different factors.

The linear (L) and quadratic contrasts (Q) for X_1 are:

$$L_{X_1}=-3\ (934.7)+(-1)\ (1008.1)+1\ (1110.2)+3\ (1193.8)=879.4$$
$$Q_{X_1}=1\ (934.7)+(-1)\ (1008.1)+(-1)\ (1110.2)+1\ (1193.8)=10.2$$

The number of observations at each level is 12, thus the number of replicates is n = 12.

The linear and quadratic coefficients for X_1 are:

$$b_1=\frac{Linear\ \ contrast\ \ for\ X_1}{20\times n}=\frac{879.4}{20\times 12}=3.664167$$
$$b_{11}=\frac{Quadratic\ \ contrast\ \ for\ X_1}{16\times n}=\frac{10.2}{16\times 12}=0.053125$$

The same formulae are used to find the linear and quadratic coefficients for X_2.

$$b_2 = \frac{Linear\ \ contrast\ \ for\ X_2}{20 \times n} = \frac{1109.6}{20 \times 12} = 4.623333$$
$$b_{22} = \frac{Quadratic\ \ contrast\ \ for\ X_2}{16 \times n} = \frac{-153.8}{16 \times 12} = -0.80104$$

The coefficient of the linear interaction between X_1 and X_2 is

$$b_{12} = \frac{Linear\ \ contrast\ \ for\ X_1X_2}{400 \times n}$$

The linear contrast for X_1X_2 is:

Similarly, the same steps can be used to find

$$L_{X_1X_2} = (-3)\ (-3)\ (153.7) + (-3)\ (-1)\ (210.8) + \ \ldots + (3)\ (1)\ (298.2)$$
$$+(3)(3)(299.4) = -1626.2$$

The number of observations at each joint level is 3 which represents the number of replicates n =3. Thus, the value of b_{12} is:

$$b_{12} = \frac{Linear\ \ contrast\ \ for\ X_1X_2}{400 \times n} = \frac{-1626.2}{400(3)} = -1.33517$$

Next step is to analyze the second experiment; Only X_3 at three levels,

The linear and quadratic coefficients for X_3 are:
The linear contrast for X_3 is:

$$L_{X_3} = (-1)(1379.2) + (0)(1416.8) + (1)(1450.8) = 71.6$$
$$Q_{X_3} = (1)(1379.2) + (-2)(1416.8) + (1)(1450.8) = -3.6$$

The number of observation at each level is 16 which represents the number of replicates n = 16.

The linear and quadratic coefficients for X_3 are:

$$b_3 = \frac{Linear\ \ contrast\ \ for\ X_3}{2n} = \frac{71.6}{2(16)} = 2.2375$$
$$b_{33} = \frac{Quadratic\ \ contrast\ \ for\ X_3}{2n} = \frac{-3.6}{2(16} = -0.1125$$

The coefficient of the linear interaction between X_1X_3 and X_2X_3 can be calculated using the following formula:

$$b_{lL} = \frac{Linear\ \ contrast\ \ for\ \ A_lA_L}{40 \times n}$$

The linear contrast for X_1 X_3 is

$$L_{X_1X_3} = (-3)(-1)(292.4) + (-3)(0)(315.5) + \ \ldots + 3(0)(397.1 + 3(1)(397.1) = -134$$

The number of observation at each joint level is 4 which represents the replicates n = 4. The value of b_{13} is:

$$b_{X_1X_3} = \frac{Linear\ \ contrast\ \ for\ \ X_1X_3}{40 \times n} = \frac{-134}{40(4)} = -0.8375$$

The same formula is used to find the interaction between X_2 and X_3 (b_{23}).

$$b_{X_2X_3} = \frac{Linear\ \ contrast\ \ for\ \ X_2X_3}{40 \times n} = \frac{73}{40(4)} = -0.45625$$

The intercept b_0 is calculated as follows:

$$b_0 = \overline{Y} - b_{11}\overline{X}_1 - b_{22}\overline{X}_2 - b_{33}\overline{X}_3$$
$$b_0 = 88.475 - 0.53125(240/48) - (-0.80104)(240/48)$$
$$-(-0.1125)(32/48) = 92.28958$$

The second-order response surface model given in equation (6) becomes:

$$Y = 92.2895 + 3.6642X_1 + 4.623X_2 + 2.2375X_3 + 0.053125X_1^2 -$$
$$0.80104X_2^2 - 0.1125X_3^2 - 1.3552X_1X_2 - 0.8375X_{13} - 0.45625X_2X_3$$

To write this equation in actual variables, the relationship between the actual and coded form is used.

$$Y = 92.2895 + 3.6642\ [x1 - 5] + 4.623\ [\frac{x2 - 0.7}{0.1}] + 2.2375\ [\frac{x3 - 4.5}{3.5}]$$
$$+0.053125\ [x1 - 5]^2 - 0.80104\ [\frac{x2 - 0.7}{0.1}]^2 - 0.1125\ [\frac{x3 - 4.5}{3.5}]^2$$
$$-1.3552\ [x1 - 5][\frac{x2 - 0.7}{0.1}] - 0.8375\ [x1 - 5][\frac{x3 - 4.5}{3.5}] - 0.45625\ [\frac{x2 - 0.7}{0.1}][\frac{x3 - 4.5}{3.5}]$$
$$Y = -56.30 + 13.696x_1 + 232.004x_2 + 2.831x_3 + 0.053x_1^2 - 80.104x_2^2 - 0.009x_3^2$$
$$-13.552x_1x_2 - 0.239x_1x_3 - 1.304x_2x_3$$

Table 4: Comparison between the proposed procedure and the least squares method for fitting a second-order model to 3^14^2 design

Parameter	Proposed procedure	Least squares method
b_0	-56.30	-56.30
b_1	13.696	13.696
b_2	232.004	232.004
b_3	2.831	2.831
b_{11}	0.053	0.053
b_{22}	-80.104	-80.104
b_{33}	-0.009	-0.009
b_{12}	-13.552	-13.552
b_{13}	-0.239	-0.239
b_{23}	-1.304	-1.304

The results of the least squares method and the proposed procedure are given in Table 4.

It can be seen from Table 4 that both methods, the least squares and the proposed procedure, gave the same results. Furthermore, the proposed procedure estimated the coefficients independently which cannot be calculated by using the least squares method.

CONCLUSION

The results showed that the new formulae for fitting a second-order model to mixed three-level and four-level designs provide fixed formulae regardless of the number of factors in the experiment as well as easy and simple formulae for statisticians and non-statisticians to avoid the complication in using the least squares method and the possibility of estimating each coefficient independently as well.

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