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A Comparative Study of Sumudu Decomposition Method and Sumudu Projected Differential Transform Method

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Abstract: In this paper, the Sumudu Decomposition Method (SDM) and Sumudu projected differential transform method (SPDTM) are used to solve the linear and nonlinear partial differential equations. In Sumudu projected differential transform method the nonlinear terms can be easily handled by using of projected differential transform method. The results shown that the (SPDTM) has an advantage over the (SDM) that it takes less time to solve the nonlinear problems without using the A domain polynomials.

Key words: Sumudu decomposition method · Sumudu projected differential transform method · Projected differential transforms method · Partial differential equations · A domain polynomials

INTRODUCTION

Nonlinearity exists everywhere and nature is nonlinear in general. The search for a better and easy to use tool for the solution of nonlinear equations that illuminate the nonlinear phenomena of real life problems of science and engineering has recently received a continuing interest various methods, therefore, were proposed to find exact or approximate solutions of nonlinear partial differential equations.

Several techniques such as A domain decomposition method [1], Variational iteration method [2, 3], Homotopy perturbation method [4], Laplace decomposition method [5, 6,11], Sumudu decomposition method [7] and Elzaki transforms method [8] and modified KdV equations [12,16] and (G'/G) expansion method [13-15,17], have been used to solve linear and nonlinear partial differential equations.

The projected differential transforms method was first proposed by Nuran Gnzel [9, 10] and successfully employed to solve nonlinear partial differential equations.

In this paper, the main objective is to introduce a comparative study of linear and nonlinear partial differential equations by using (SDM) and (SPDTM).

Basic Idea of Sumudu Projected Differential Transform Method (SPDTM)

Projected Differential Transform Method : In this section we introduce the projected differential transform method which is modified of the differential transform method.

Definition: The basic definition of projected differential transforms method of function $f(x_1, x_2, ..., x_n)$ is defined as

$$f\left(x_1, x_2, ..., x_{n-1}, k\right) = \frac{1}{k!}\left[\frac{\partial^k f\left(x_1, x_2, ..., x_n\right)}{\partial x_n{}^k}\right]_{x_n = x_0} \quad (1)$$

Such that $f(x_1, x_2, ..., x_n)$ is the original function and $f\left(x_1, x_2, ..., x_{n-1}, k\right)$ is projected transform method.

And the differential inverse transform of $f\left(x_1, x_2, ..., x_{n-1}, k\right)$ is defined as

$$f\left(x_1, x_2, ..., x_n\right) = \sum_{k=0}^{\infty} f\left(x_1, x_2, ..., x_n\right)\left(x - x_0\right)^k \quad (2)$$

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The fundamental theorems of the projected differential transform are

Theorems:

(1) If

$$z(x_1,x_2,\ldots,x_n)=u(x_1,x_2,\ldots,x_n)\pm v(x_1,x_2,\ldots,x_n)$$

then

$$z(x_1,x_2,\ldots,x_{n-1},k)=u(x_1,x_2,\ldots,x_{n-1},k)\pm v(x_1,x_2,\ldots,x_{n-1},k)$$

(2) If

$$z(x_1,x_2,\ldots,x_n)=c\,u(x_1,x_2,\ldots,x_n)$$

then

$$z(x_1,x_2,\ldots,x_{n-1},k)=c\,u(x_1,x_2,\ldots,x_{n-1},k)$$

(3) If

$$z(x_1,x_2,\ldots,x_n)=\frac{d\,u(x_1,x_2,\ldots,x_n)}{d\,x_n}$$

then

$$z(x_1,x_2,\ldots,x_{n-1},k)=(k+1)u(x_1,x_2,\ldots,x_{n-1},k+1)$$

(4)If

$$z(x_1,x_2,\ldots,x_n)=\frac{d^n u(x_1,x_2,\ldots,x_n)}{d\,x_n{}^n}$$

then

$$z(x_1,x_2,\ldots,x_{n-1},k)=\frac{(k+n)}{k!}u(x_1,x_2,\ldots,x_{n-1},k+n)$$

(5) If

$$z(x_1,x_2,\ldots,x_n)=u(x_1,x_2,\ldots,x_n)\,v(x_1,x_2,\ldots,x_n)$$

then

$$z(x_1,x_2,\ldots,x_{n-1},k)=\sum_{m=0}^{k}u(x_1,x_2,\ldots,x_{n-1},m)\,v(x_1,x_2,\ldots,x_{n-1},k-m)$$

(6) If

$$z(x_1,x_2,\ldots,x_n)=x_1^{q_1}\,x_2^{q_2}\ldots\ldots x_n^{q_n}$$

then

$$z(x_1,x_2,\ldots,x_{n-1},k)=\delta(x_1,x_2,\ldots,x_{n-1},q_n-k)=\begin{cases}1 & k=q_n\\ 0 & k\neq q_n\end{cases}$$

Note that c is constant and n is a nonnegative integer.

MATERIALS AND METHODS

To mention the basic idea of this method, we consider a general nonlinear non- homogeneous partial differential equation with initial conditions of the form

$$\begin{aligned}&DU(x,t)+RU(x,t)+NU(x,t)=g(x,t)\\&U(x,0)=h(x)\;,U_t(x,0)=f(x)\end{aligned}\tag{3}$$

Where D is the second order linear differential operator $D=\frac{\partial^2}{\partial t^2}$, is linear differential operator of less order than D, N represent the general nonlinear operator and $g(x,t)$ is the source term.

Taking the Sumudu transform of both sides of Eq. (3), we get

$$S[DU(x,t)]+S[RU(x,t)]+S[N(x,t)]=S[g(x,t)]\tag{4}$$

Using the differentiation property of the sumudu transform and given initial conditions, we have

$$\begin{aligned}&S[U(x,t)]=u^2S[g(x,t)]+h(x)+\\&u\,f(x)-u^2S[RU(x,t)+NU(x,t)].\end{aligned}\tag{5}$$

Now, applying the inverseoff Sumudu transform to both sides of Eq. (5), we get

$$U(x,t)=G(x,t)-S^{-1}\left[u^2S[RU(x,t)+NU(x,t)]\right]\tag{6}$$

Where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now, we apply the projected differential transform method, the recursive relation is given by

$$\begin{aligned}&U_0(x,t)=G(x,t),\\&U_{m+1}(x,t)=-S^{-1}\left[u^2S[A_m+B_m]\right].\end{aligned}\tag{7}$$

Where A_m and are t.ted differential transform of $RU(x,t)$ and $NU(x,t)$.

From equation (7), we have:

$$\begin{aligned}&U_0(x,t)=G(x,t),\\&U_1(x,t)=-S^{-1}\left[u^2S[A_0+B_0]\right],\\&U_2(x,t)=-S^{-1}\left[u^2S[A_1+B_1]\right],\\&U_3(x,t)=-S^{-1}\left[u^2S[A_2+B_2]\right].\end{aligned}\tag{8}$$

And so on. Then the solution of equation (3) is

$$U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + ... \qquad (9)$$

Example 2.1 : Consider the linear partial differential equation

$$U_{xx} + U_{tt} = 0 \qquad (10)$$

The initial condition

$$U(x,0) = 0 \ , U_t(x,0) = \cos x \qquad (11)$$

Taking the Sumudu transform of both sides of Eq. (11) and making use of the initial condition to obtain

$$S\left[U(x,t)\right] = u\cos x - u^2 S\left[U_{xx}\right]. \qquad (12)$$

Applying the inverse Sumudu transform implies that

$$U(x,t) = t\cos x - S^{-1}\left[u^2 S\left[U_{xx}\right]\right]. \qquad (13)$$

Using the projected differential method, this leads to the recursive relation

$$\begin{aligned} U_0(x,t) &= t\cos x, \\ U_{m+1}(x,t) &= -S^{-1}\left[u^2 S\left[A_m\right]\right]. \end{aligned} \qquad (14)$$

Where $A_m = \frac{\partial^2 U(x,m)}{\partial x^2}$ s projected differential transform of $\frac{\partial^2 U(x,t)}{\partial x^2}$

This gives

$$\begin{aligned} U_0(x,t) &= t\cos x, \\ U_1(x,t) &= -S^{-1}\left[u^2 S[A_0]\right] = -S^{-1}\left[u^2 S[-t\cos x]\right] = \frac{t^3}{3!}\cos x, \\ U_2(x,t) &= -S^{-1}\left[u^2 S[A_1]\right] = -S^{-1}\left[u^2 S\left[-\frac{t^3}{3!}\cos x\right]\right] = \frac{t^5}{5!}\cos x. \end{aligned} \qquad (15)$$

And so on. The solution in the series form is given by

$$U(x,t) = \cos x\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + ...\right),. \qquad (16)$$

The closed form solution is given by

$$U(x,t) = \cos x \sinh t. \qquad (17)$$

Example 2.2: Consider the nonlinear partial differential equation

$$U_t = U_x^{\ 2} + U U_{xx} \qquad (18)$$

With the initial condition

$$U(x,0) = x^2 \qquad (19)$$

Following the analysis presented above gives

$$U(x,t) = x^2 + S^{-1}\left[u\, S\left[U^2_{\ x} + U U_{xx}\right]\right]. \qquad (20)$$

The recursive relation

$$\begin{aligned} U_0(x,t) &= x^2, \\ U_{m+1}(x,t) &= S^{-1}\left[u\, S\left[A_m + B_m\right]\right]. \end{aligned} \qquad (21)$$

Where

$$A_m = \sum_{m=0}^{h} \frac{\partial U(x,m)}{\partial x}\frac{\partial U(x,h-m)}{\partial x},$$

$$B_m = \sum_{m=0}^{h} U(x,m) U_{xx}(x,h-m)$$

Are projected differential transform of $U^2{}_x$ and UU_{xx}

This gives

$$\begin{aligned} U_0(x,t) &= x^2, \\ U_1(x,t) &= S^{-1}\left[u\,S[A_0+B_0]\right] = S^{-1}\left[uS\left[4x^2+2x^2\right]\right] = 6x^2 t, \\ U_2(x,t) &= S^{-1}\left[u\,S[A_1+B_1]\right] = S^{-1}\left[uS\left[48x^2 t + 24x^2 t\right]\right] = 36x^2 t^2. \end{aligned} \qquad (22)$$

And so on. The solution in the series form is given by

$$U(x,t) = x^2\left(1 + 6t + 36t^2 + ...\right), \qquad (23)$$

The closed form solution is given by

$$U(x,t) = \frac{x^2}{1-6t}. \qquad (24)$$

Basic Idea of Sumudu Decomposition Method (SDM): To mention the basic idea of this method, we consider a general nonlinear non- homogeneous partial differential equation with initial conditions of the form

$$\begin{aligned} DU(x,t) + RU(x,t) + NU(x,t) &= g(x,t) \\ U(x,0) = h(x) \ ,U_t(x,0) &= f(x) \end{aligned} \qquad (25)$$

Where D is the second order linear differential operator $D=\frac{\partial^2}{\partial t^2}$, is linear differential operator of less order than D, N represent the general nonlinear operator and $g(x,t)$ is the source term.

Taking the Sumudu transform of both sides of Eq. (25), we get

$$S\left[DU(x,t)\right]+S\left[RU(x,t)\right]+S\left[N(x,t)\right]=S\left[g(x,t)\right] \tag{26}$$

Using the differentiation property of the Sumudu transform and given initial conditions, we have

$$\begin{aligned} S\left[U(x,t)\right] &= u^2 S\left[g(x,t)\right] \\ &+h(x)+u\,f(x)-u^2 S\left[RU(x,t)+NU(x,t)\right]. \end{aligned} \tag{27}$$

Now, applying the inverse Sumudu transform to both sides of Eq. (27), we get

$$U(x,t)=G(x,t)-S^{-1}\left[u^2 S\left[RU(x,t)+NU(x,t)\right]\right] \tag{28}$$

Where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now, we apply the A domain decomposition method

$$U(x,t)=\sum_{n=0}^{\infty} U_n(x,t) \tag{29}$$

And the nonlinear term can be decomposed as

$$NU(x,t)=\sum_{n=0}^{\infty} A_n(U) \tag{30}$$

For some A domain polynomials $A_n(U)$ that are given by

$$\begin{aligned} &A_n\left(U_0,U_1,U_2,......,U_n\right) \\ &=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{n=0}^{\infty}\lambda^n U_n\right)\right]_{\lambda=0}, \quad n=0,1,2,.... \end{aligned} \tag{31}$$

Substituting Eq. (29) and Eq. (30) into Eq. (28), we get

$$\sum_{n=0}^{\infty} U_n(x,t)=G(x,t)-S^{-1}\left[u^2 S\left[R\sum_{n=0}^{\infty}U_n(x,t)+\sum_{n=0}^{\infty}A_n(U)\right]\right] \tag{32}$$

So that the recursive relation is given by

$$\begin{aligned} U_0(x,t) &= G(x,t), \\ U_{k+1}(x,t) &= -S^{-1}\left[u^2 S\left[RU_k + A_k\right]\right].\ k\geq 0. \end{aligned} \tag{33}$$

Example 3.1: Consider the linear partial differential equation

$$U_{xx}+U_{tt}=0 \tag{34}$$

With the initial conditions

$$U(x,0)=0\ ,U_t(x,0)=\cos x \tag{35}$$

Taking the Sumudu transform of both sides of Eq. (34) and making use of the initial condition to obtain

$$S\left[U(x,t)\right]=u\cos x-u^2 S\left[U_{xx}\right]. \tag{36}$$

Applying the inverse Sumudu transform implies that

$$U(x,t)=t\cos x-S^{-1}\left[u^2 S\left[U_{xx}\right]\right]. \tag{37}$$

Using the decomposition series for the linear term $U(x,t)$ gives

$$\sum_{n=0}^{\infty}U_n(x,t)=t\cos x-S^{-1}\left[u^2 S\left(\sum_{n=0}^{\infty}(U_n)_{xx}(x,t)\right)\right]. \tag{38}$$

This leads to the recursive relation

$$\begin{aligned} U_0(x,t) &= t\cos x, \\ U_{k+1}(x,t) &= -S^{-1}\left[u^2 S\left[(U_k)_{xx}\right]\right], k\geq 0. \end{aligned} \tag{39}$$

This gives

$$\begin{aligned} U_0(x,t) &= t\cos x, \\ U_1(x,t) &= -S^{-1}\left[u^2 S\left[(U_0)_{xx}\right]\right]=\frac{t^3}{3!}\cos x, \\ U_2(x,t) &= -S^{-1}\left[u^2 S\left[(U_1)_{xx}\right]\right]=\frac{t^5}{5!}\cos x. \end{aligned} \tag{40}$$

And so on. The solution in the series form is given by

$$U(x,t)=\cos x\left(t+\frac{t^3}{3!}+\frac{t^5}{5!}+......\right), \tag{41}$$

The closed form solution is given by

$$U(x,t) = \cos x \sinh t. \tag{42}$$

Example 3.2: Consider the nonlinear partial differential equation

$$U_t = U_x^{\,2} + U U_{xx} \tag{43}$$

With the initial condition

$$U(x,0) = x^2 \tag{44}$$

Following the analysis presented above gives

$$\sum_{n=0}^{\infty} U_n(x,t) = x^2 + S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} B_n\right]\right]. \tag{45}$$

The recursive relation

$$\begin{aligned} U_0(x,t) &= x^2, \\ U_{k+1}(x,t) &= S^{-1}\left[u\,S\left[A_k + B_k\right]\right], k \geq 0. \end{aligned} \tag{46}$$

The A domain polynomials for the nonlinear term U^2_x have been derived in the form

$$A_0 = U^2_{0_x},$$
$$A_1 = 2U_{0_x} U_{1_x}.$$

And The A domain polynomials for the nonlinear term UU_{xx} have been derived in the form

$$B_0 = U_0 U_{0_{xx}},$$
$$B_1 = U_1 U_{0_{xx}} + U_0 U_{1_{xx}}.$$

The first few components can be identified by

$$\begin{aligned} U_0(x,t) &= x^2, \\ U_1(x,t) &= S^{-1}\left[u\,S\left[A_0 + B_0\right]\right] = 6x^2 t, \\ U_2(x,t) &= S^{-1}\left[u\,S\left[A_1 + B_1\right]\right] = 36\,x^2 t^2. \end{aligned} \tag{47}$$

And so on. The solution in the series form is given by

$$U(x,t) = x^2\left(1 + 6t + 36t^2 + \ldots\ldots\right),$$

The closed form solution is given by

$$U(x,t) = \frac{x^2}{1-6t}.$$

CONCLUSION

In the present paper, (SDM) is employed for solving linear and nonlinear partial differential equations; the same problems are solved by (SPDTM). It is worth mentioning that the (SPDTM) is a very simple technique to handle linear and nonlinear partial differential equations than the (SDM) and also the obtained series solution by (SPDTM) converges faster than those obtained by (SDM). The results reveal that the (SPDTM) is a powerful technique and can be applied to other applications.

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