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Strength and Stability of Orthotropic Shells

Vladimir Karpov and Alexej Semenov

Department of Applied Mathematics and Information Technology,
Construction Faculty, St. Petersburg State University of Architecture and
Civil Engineering, Vtoraja Krasnoarmejskaja Ul. 4, St. Petersburg, 190005, Russia

Abstract: The work investigates thin walled orthotropic shells. Their deformation model is built taking into account the geometrical non-linearity and transverse shifts. The investigation algorithm of this model is based on the Ritz method and the method of continuing solution employing the best parameter. This enables the study of strength and stability of the shells under consideration. For analysis of special points (upper and lower critical loads, bifurcation points) the Jacobian matrix of linear equation systems is used. Based on the developed algorithm, a software product is made. A study was carried out on the strength and stability of some versions of sloping shells from different types of carbon fiber-reinforced plastic. It has been shown that the loss of stability of thin-walled shells under consideration occurs before the strength is lost. The loss of strength in different shells takes place in different components of maximum tension. It was established that in case of a geometrically non-linear approach, the critical loads in the event of strength reduction are essentially lower than in case of the linear computing method.

Key words: Shells · Strength · Stability · Mathematical model · Algorithm · Software product · Orthotropicity of material

INTRODUCTION

With the emergence of new construction materials (carbon fiber-reinforced plastics, boron-fiber reinforced plastics, glass-reinforced plastics, etc.) which can be regarded as orthotropic [1-5] it became acutely important to know the strength and stability of structure from such materials [6-7]. The optimum, in terms of weight and strength, are shelled structures featuring sufficiently high rigidity and a variety of structural shapes [3, 8].

The shelled structures are used in different fields of engineering, including construction, for covering long-span building arrangements [9, 10]. The thin-shelled structures have a single shortcoming, propensity to stability reduction. The study on the stability of thin-walled shelled structures shows the necessity to solve many complex non-linear differential equations in partial derivatives. Therefore a solution to these problems is only possible with the use of modern computing machines [11, 12].

In order to obtain more precise results, it is necessary to take into account all the factors affecting the tension-deformed state of the shells, i.e. to develop the most precise model of their deformation. After that, an optimum computing algorithm must be developed for investigating those models and creating a software product [13].

MATERIALS AND METHODS

The shell's mathematical deformation model is formed as a function of the shell deformation's full energy. To minimize this functional, the Ritz method is used. For converting algebraic non-linear equations to a sequence of solutions applied to an array of algebraic linear equations, a method is used known as continuing solution based on the best parameter [15]. For solving an array of algebraic linear equations the Gaussian method is used. The software product is developed on the basis of Maple 17 analytical calculations with the use of paralleling calculation technique [16].

Corresponding Author: Vladimir Karpov, Department of Applied Mathematics and Information Technology, Construction Faculty, St. Petersburg State University of Architecture and Civil Engineering, Vtoraja Krasnoarmejskaja ul. 4, St. Petersburg, 190005, Russia. Tel: +7(812)5750549.

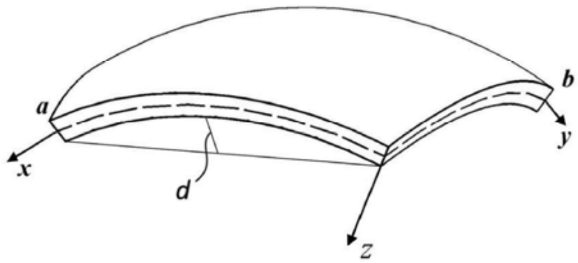

Fig. 1: The shell's local coordinate system

The Main Part: Considered are thin-walled resilient shells fixed by hinges along the outline and sustaining an evenly distributed transverse load q. The accepted system of local orthogonal curve coordinates is shown in Fig. 1.

The mathematical model of the thin-walled shell's deformation consists of geometrical ratios in the median surface of the shell:

$$\varepsilon_x = \frac{1}{A}\frac{\partial U}{\partial x} + \frac{1}{AB}V\frac{\partial A}{\partial y} - k_x W + \frac{1}{2}\theta_1^2,$$
$$\varepsilon_y = \frac{1}{B}\frac{\partial V}{\partial y} + \frac{1}{AB}U\frac{\partial B}{\partial x} - k_y W + \frac{1}{2}\theta_2^2,$$
$$\gamma_{xy} = \frac{1}{A}\frac{\partial V}{\partial x} + \frac{1}{B}\frac{\partial U}{\partial y} - \frac{1}{AB}U\frac{\partial A}{\partial y} - \frac{1}{AB}V\frac{\partial B}{\partial x} + \theta_1\theta_2,$$
$$\theta_1 = -\left(\frac{1}{A}\frac{\partial W}{\partial x} + k_x U\right), \quad \theta_2 = -\left(\frac{1}{B}\frac{\partial W}{\partial y} + k_y V\right), \tag{1}$$

where ε_x, ε_y are deformations of extension along the x, y coordinates of the median surface; γ_{xy} is deformation of the shift in XOY plane; k_x, k_y are principal curves of the shell on x and y axes; A, B are Lamé parameters characterizing the shell geometry; $U = U(x, y)$ $V = V(x, y)$, $W = W(x, y)$ are functions of the median surface point shifts on, respectively, x, y and z coordinate axes. In the layer, distanced on z from the median surface, the deformation assumes the aspect:

$$\varepsilon_x^z = \varepsilon_x + z\,\chi_1; \;\; \varepsilon_y^z = \varepsilon_y + z\,\chi_2;$$
$$\gamma_{xy}^z = \gamma_{xy} + 2\,z\,\chi_{12},$$

where the functions of change of curves χ_1, χ_2 and of torsion χ_{12} assume the aspect:

$$\chi_1 = \frac{1}{A}\frac{\partial \Psi_x}{\partial x} + \frac{1}{AB}\frac{\partial A}{\partial y}\Psi_y; \quad \chi_2 = \frac{1}{B}\frac{\partial \Psi_y}{\partial y} + \frac{1}{AB}\frac{\partial B}{\partial x}\Psi_x,$$
$$2\chi_{12} = \frac{1}{A}\frac{\partial \Psi_y}{\partial x} + \frac{1}{B}\frac{\partial \Psi_x}{\partial y} - \frac{1}{AB}\left(\frac{\partial A}{\partial y}\Psi_x + \frac{\partial B}{\partial x}\Psi_y\right).$$

Here $\Psi_x = \Psi_x\,(x, y)$, $\Psi_y = \Psi_y\,(x, y)$ are functions characterizing the normal torsion angles in planes XOZ, YOZ. In addition, if the transverse shifts (Timoshenko - Reissner model) are considered then:

$$\gamma_{xz} = k\;f(z)\,(\psi_x - \theta_1);$$
$$\gamma_{yz} = k\;f(z)\,(\psi_y - \theta_2).$$

Here $f(z)$ is a function characterizing the distribution of tensions τ_{xz}, τ_{yz} over the shell thickness. At

$$f(z) = 6\left(\frac{1}{4} - \frac{z^2}{h^2}\right) \quad k = \frac{5}{6}.$$

physical ratios of orthotropic shells are built based on Hook's generalizing law, assuming the following form:

$$\sigma_x = \frac{E_1}{1-\mu_1\mu_2}\left[\varepsilon_x + \mu_2\varepsilon_y + z\left(\chi_1 + \mu_2\chi_2\right)\right];$$
$$\sigma_y = \frac{E_2}{1-\mu_1\mu_2}\left[\varepsilon_y + \mu_1\varepsilon_x + z\left(\chi_2 + \mu_1\chi_1\right)\right];$$
$$\tau_{xy} = G_{12}\left[\gamma_{xy} + 2z\chi_{12}\right];$$
$$\tau_{xz} = G_{13}kf(z)\left(\psi_x - \theta_1\right); \; \tau_{yz} = G_{23}kf(z)\left(\psi_y - \theta_2\right), \tag{2}$$

In the said ratios, E_1, E_2 are, respectively, elastic modules in x and y directions, μ_1, μ_2 are Poisson ratio and G_{12}, G_{13}, G_{23} are shift modules in XOY, XOZ, YOZ planes.

Integrating the tensions (2) on z in the limits of $-\frac{h}{2}$ to $\frac{h}{2}$ we obtain tensions and moments for a unit of section length represented on the coordinate surface:

$$N_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x dz; \quad N_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y dz; \quad N_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} dz;$$
$$M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_x dz; \quad M_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_y dz; \quad M_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\tau_{xy} dz;$$
$$Q_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} dz; \quad Q_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yz} dz. \tag{3}$$

The functional of the shell's complete deformation energy will have following form:

$$E_p = \frac{1}{2}\int_0^a\int_0^b \{N_x\varepsilon_x + N_y\varepsilon_y + \frac{1}{2}\left(N_{xy} + N_{yx}\right)\gamma_{xy} + M_x\chi_1 + M_y\chi_2 +$$
$$+\left(M_{xy} + M_{yx}\right)\chi_{12} + Q_x\left(\psi_x - \theta_1\right) + Q_y\left(\psi_y - \theta_2\right) - 2qW\}ABdxdy. \quad (4)$$

This functional, together with boundary conditions, forms a mathematical model of the shell deformation. The mathematical model takes into account the geometrical non-linearity, transverse shifts and the material orthotropy.

To minimize this functional, the Ritz method is used in which case the unknown functions will be presented as follows:

$$U(x,y) = \sum_{I=1}^{N} U(I)Z1(I); V(x,y) = \sum_{I=1}^{N} V(I)Z2(I);$$
$$W(x,y) = \sum_{I=1}^{N} W(I)Z3(I); \psi_x(x,y) = \sum_{I=1}^{N} PS(I)Z4(I);$$
$$\psi_y(x,y) = \sum_{I=1}^{N} PN(I)Z5(I), \quad (5)$$

where *U(I), V(I), W(I), PS(I), PN(I)* are unknown numeric parameters. Z1*(I)* – Z(5*(I)* are known approximating functions of arguments x and y that meet the set boundary conditions on the shell outline.

Substituting (5) for (4) we shall find derivatives of the functional based on unknown numeric parameters and then will equate them to zero. As a result, we shall get a system of non-linear algebraic equations which can be briefly presented as follows:

$$F(X, P) = 0, \quad (6)$$

where X is a vector of unknown numeric parameters and P is a load.

As the best parameters for continuing the solution it is proposed to use the arc length λ as a curve for a multiple solutions. Thus the load parameter P is characteristically equal to the rest of unknown quantities and it is convenient to add it to other parameters.

$$\tilde{X} = (X,P)^T = \left(U(I),V(I),W(I),PS(I),PN(I),P\right)^T, I = 1..N$$

The parameter λ is obviously not a part of the equations system (5) and is linked to the variables of the problem from $\tilde{X}$ vector in the following way:

$$(d\lambda)^2 = \sum_{I=1}^{N}\left[(dU(I))^2 + (dV(I))^2 + (dW(I))^2 + (dPS(I))^2 + (dPN(I))^2 + (dP)^2\right]. \quad (7)$$

The larger is the difference between vectors $\frac{d\tilde{X}}{d\lambda}$ at the current and previous stage, the worse is the system matrix's determination and hence the less stable is the computing procedure. Therefore, in computing it makes sense to choose the network adaptively.

The method of continuing solution based on the best parameter with adaptive choice of the network makes it possible to find the upper and lower critical loads as well as bifurcation points.

On the basis of the above given algorithm a software product was developed [16] which enables investigating the strength and stability of various types of orthotropic shells. Considered as an example are rectangular sloping shells whose numeric parameters are shown in Table 1.

Mechanical parameters of materials of structures under consideration are given in Table 2.

Also, Table 2 shows maximum values of tensions F^+_1, F^-_1, F^+_2, F^-_2 on x and y axes (respectively, indexes 1 and 2) and shifting F_{12} in XOY plane. The sign "+" corresponds to extension, the sign "-" to compression.

For orthotropic shells the strength is evaluated by comparing the values of all tension values with the maximum values of tension resulting from extension and compression. The subsequently obtained values are not used practically but can be scientifically interesting which is why the structures were also computed in the supercritical condition.

Let us build some characteristic points' equilibrium state curve by increasing gradually the load values. The maximum points of this curve correspond to the moments of stability loss while the load, corresponding to these points, is critical. The curve points of equilibrium states in which Jacobian matrixes of the system of algebraic linear equations det(J) turn unto zero are special points. These points may correspond to upper and lower critical loads as well as to bifurcation points.

Fig. 2 presents the "load - W deflection" chart for version 1 shell be made from T300/976 carbon fiber-reinforced plastic. The points in Fig. 2 denote the load value at which the extension's maximum tension value along the axis is reached. Thus, for the given shell the stability will be lost before strength will be.

Here and further on in the charts the black color shows the maximum deflection curve; the red shows the deflection curve in the middle of the structure $\left(x = \frac{a}{2}, y = \frac{b}{2}\right)$; the blue shows in the quarter $\left(x = \frac{a}{4}, y = \frac{b}{4}\right)$

Table 1: Geometric parameters of structures

Parameter	Version 1	Version 2	Version 3
Linear size $a = b$, (m)	5.4	54	54
Main curve radiuses $R = R_1 = R_2$, (m)	20.25	67.95	34
Shell thickness h, (m)	0.09	0.09	0.09
Summarized curve parameter k_ξ	16	477	953

Table 2: Mechanical properties of carbon fiber-reinforced plastic

	Carbon fiber-reinforced plastic		
Property	LU-P/ENFB	M60J/Epoxy	T300/976
E_1 (MPa)	$1.4.10^5$	$3.3.10^5$	$1.4.10^5$
μ_1	0.3	0.32	0.29
E_2 (MPa)	$0.97.10^4$	$0.59.10^4$	$0.97.10^4$
G_{12} (MPa)	$0.46.10^4$	$0.39.10^4$	$0.55.10^4$
G_{13} (MPa)	$0.46.10^4$	$0.39.10^4$	$0.55.10^4$
G_{23} (MPa)	$0.46.10^4$	$0.39.10^4$	$0.33.10^4$
F^+_1 (MPa)	700	1760	1517
F^-_1 (MPa)	-600	-780	-1599
F^+_2 (MPa)	27	30	46
F^-_2 (MPa)	-184	-168	-253
F_{12} (MPa)	55	39	41.4

Table 3: Values of critical loads in stability loss and critical values of strength loss

		Linear version		Non-linear version			
Shell	Carbon fiber-reinforced plastics	q_{lin} (MПa)	F	q_{nlin} (MПa)	F	q_{kr} (MПa)	$\frac{q_{lin}}{q_{nlin}}$
Version 1 $k_\xi = 16$	LU-P/ENFB	1.290	F_1^-	0.766*	F_1^-	0.771	1.68
	M60J/Epoxy	1.520	F_1^-	1.475	F_1^-	1.571	1.03
	T300/976	3.454	F_1^-	0.313*	F_2^+	0.777	11.04
Version 2 $k_\xi = 477$	LU-P/ENFB	0.253	F_1^-	0.043*	F_2^+	0.076	5.83
	M60J/Epoxy	0.297	F_1^-	0.057*	F_1^-	0.074	5.17
	T300/976	0.679	F_1^-	0.063*	F_{12}	0.078	10.8
Version 3 $k_\xi = 953$	LU-P/ENFB	0.504	F_1^-	0.406	F_1^-	0.569	1.24
	M60J/Epoxy	0.589	F_1^-	0.413	F_1^-	0.521	1.42
	T300/976	1.350	F_1^-	0.561*	F_2^+	0.586	2.41

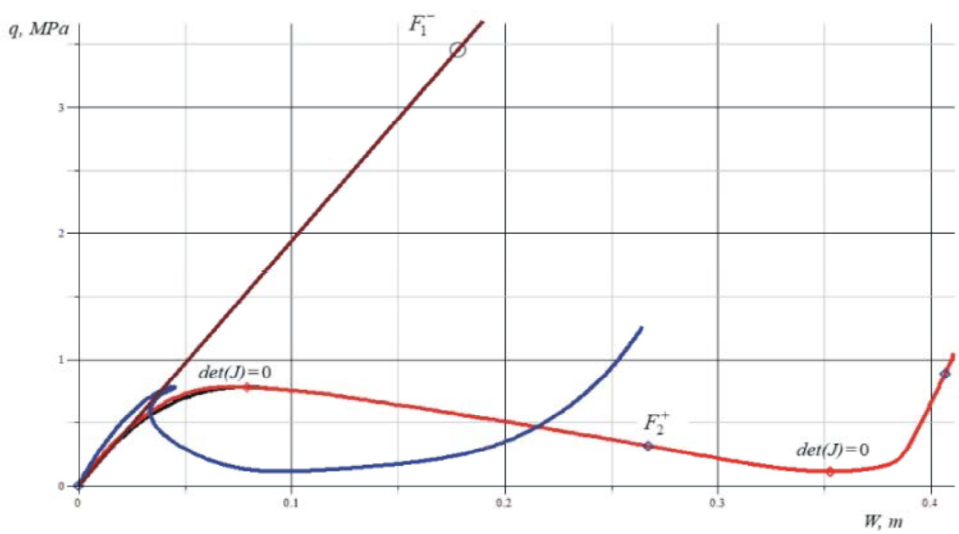

Fig. 2: "Load - deflection W" chart for the shell version 1

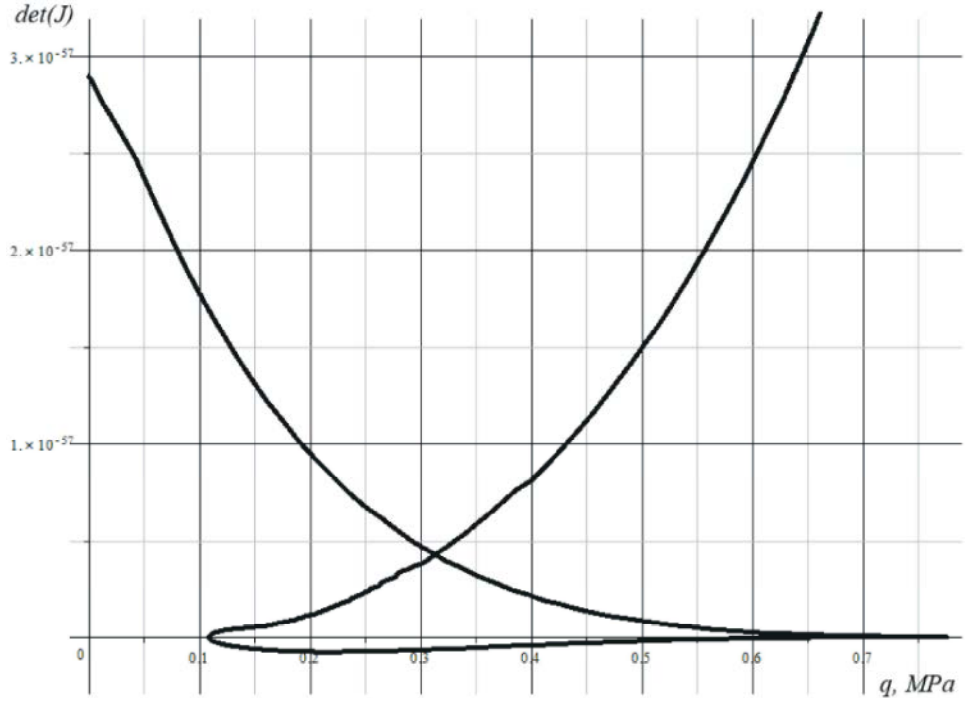

Fig. 3: *J* Matrix chart of Jacobian dependence on load values for version 1 shell

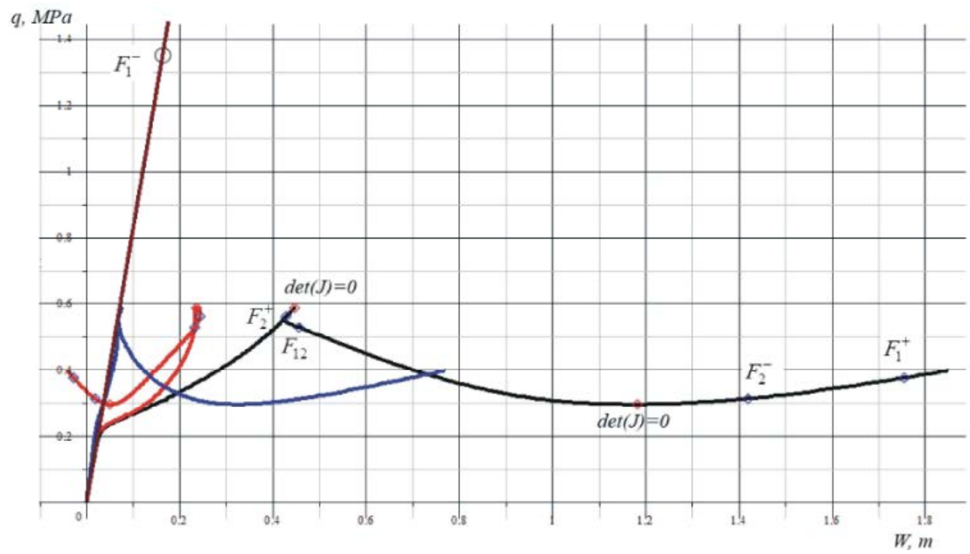

Fig. 4: "Load -deflection *W*" chart for the shell version 3

Fig. 3 shows the chart of J matrix Jacobian dependence on load value. The loads at which Jacobian turns into zero correspond to the upper and lower critical values of the shell under consideration.

Figures 4 and 5 show similar results for version 3 shell made from the same material. For the given shell the strength is also lost after the loss of stability upon attaining the maximum extension tension value on the c axis.

In Figures 2 and 4 the points on the direct line correspond to the geometrically linear solution. They show that the critical load, once the strength is lost, is much higher than that found in computing the geometrical non-linearity.

Table 3 presents the values of critical loads of stability loss and critical values of strength loss for linear and non-linear computing versions with indication of components with which the tensions exceeded the permissible maximum.

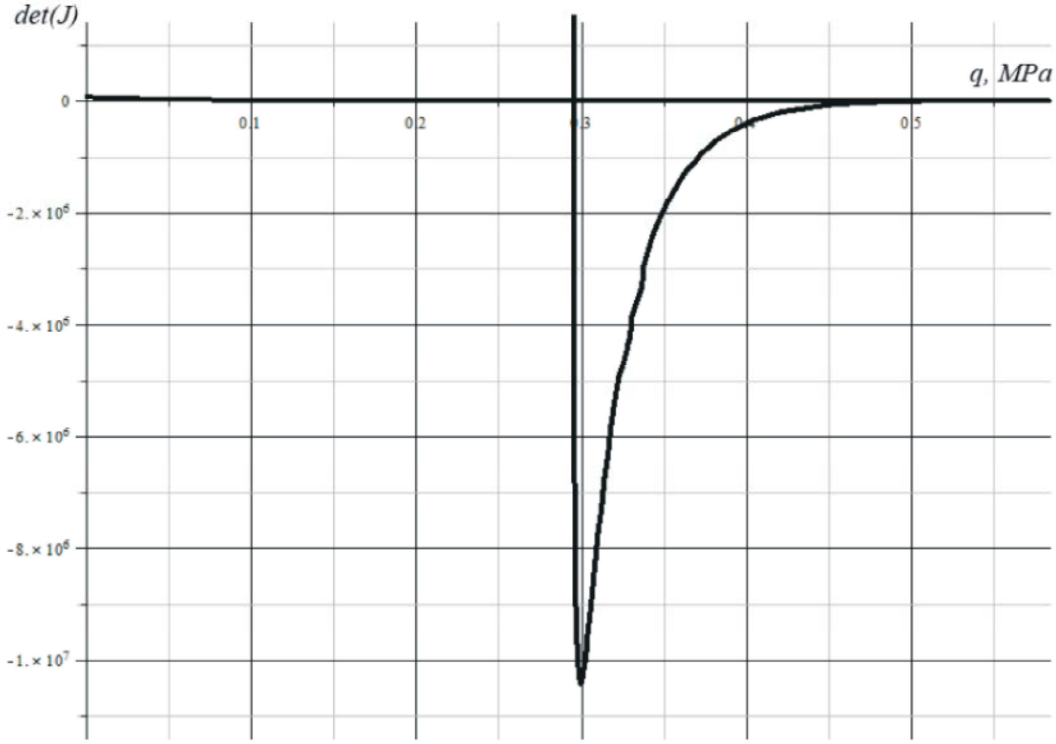

Fig. 5: *J* Matrix chart of Jacobian dependence on load values the shell version 3

The values of critical loads during strength loss in non-linear computation marked with "*" were obtained in supercritical area (after the loss of stability, on the descending branch of the "load - deflection W" chart).

CONCLUSIONS

The results of investigations showed that:

- The developed software on the basis of the most precise mathematical model of orthotropic shells' deformation and of algorithm enabling the finding of upper and lower critical loads and bifurcation points makes it possible to study the strength, stability and supercritical behavior of orthotropic shells.
- In investigating the strength of orthotropic shells it is necessary to take into account the geometrical non-linearity, otherwise the maximum permissible loads will be significantly overrated.
- In thin-walled shells the stability starts to deteriorate before the strength does.
- The nature of strength reduction is different in different materials and different shells that is why it is necessary to carry out a separate computation for each particular structure.

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