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The Flows Past a Rotating Disk by Optimal Homotopy Asymptotic Method

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Abstract: In this paper Optimal Homotopy Asymptotic Method (OHAM) has been used to the flow past a rotating disk. This method is applied to the problem obtained from flow past a rotating disk shows its efficiency and effectiveness. This method control the convergence of approximate solution in a convenient way, when it is compared with other methods of solution found in the literature as Homotopy Perturbation Method (HPM). The results shows that OHAM is simpler easier, effective and explicit.

Key words: OHAM · HPM · Rotating disk

INTRODUCTION

The flow past a rotating disk is very important in fluid dynamics. The flow is characterized by a system of three coupled non-linear ordinary differential equations. Von Karman has solved the problem by integral approach for the first time [1]. Cochran [2] solved the problem by a mixing the numerical and approximate technique. Ackroyd [3] expended the Cochran's solution for larger distances from the disk. Jain [4] find its solution by Collocation method. On the other hand Ariel [5] proposed its solution by using one parameter scale factor. Also Ariel [6] solved the problem by using HPM to get better results than one parameter scale factor.

Recently, Vasile Marinca *et al* presented Optimal Homotopy Asymptotic Method (OHAM) [7-12] for the solution of nonlinear problems. OHAM has been proved to be useful in finding the solution of nonlinear differential equations [13-18].

Our endeavor in this paper is to apply OHAM for the solution three coupled non-linear ordinary differential equations arising due to the flow past a rotating disk. In this work, it has been proved that OHAM is useful and reliable for solution of three coupled non-linear ordinary differential equations BVP.

In the succeeding section, the basic idea of OHAM [7-12] is formulated for the solution of BVPs. In Section 3, the effectiveness of the formulation of OHAM for three coupled non-linear ordinary differential equations BVP has been studied.

Basic Mathematical Theory of Oham: Let us see the OHAM to the general nonlinear BVP.

$$\mathcal{L}\big(f(x)\big) + h(x) + \mathcal{N}\big(f(x)\big) = 0, \tag{2.1}$$

with boundary conditions:

$$\mathcal{B}\left(f, \frac{df}{dx}\right) = 0. \tag{2.2}$$

where L is the linear operator, $f(x)$ is an unknown function, $h(x)$ is a known function, $N(f(x)$ is a nonlinear differential operator and B is a boundary operator.

According to OHAM, one can construct an optimal homotopy $\kappa(x, r)$: $\Phi \times [0, 1] \rightarrow \Re$ which satisfies:

$$(1-r)\Big[\mathcal{L}\big(\kappa(x,r)\big) + h(x)\Big] = H(r)\begin{bmatrix}\mathcal{L}\big(\kappa(x,r)\big) + h(x) \\ +\mathcal{N}\big(\kappa(x,r)\big)\end{bmatrix} \tag{2.3}$$

$$\mathcal{B}\left(\kappa(x,r), \frac{\partial\kappa(x,r)}{\partial x}\right) = 0, \tag{2.4}$$

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where $r \in [0,1]$ is an embedding parameter, $\kappa(x, r)$ is an unknown function, $H(r)$ is a nonzero auxiliary function. The auxiliary function, $H(r)$ is nonzero for $r \neq 0$ and $H(0) = 0$. Eq. (2.3) is the mathematical structure of OHAM homotopy.

It is defined that:

$$r = 0 \Rightarrow \kappa(x, 0) = f_0(x), \tag{2.5}$$

$$\text{and } r = 1 \Rightarrow \kappa(x,1) = f(x) \tag{2.6}$$

respectively. r varies from 0 to 1, $\kappa(x, r)$ varies from $f_0(x)$ to $f(x)$, where $f_0(x)$ is obtained from Eq. (2.1) and Eq. (2.2) for $r = 0$.

$$\mathcal{L}\big(f_0(x)\big) + h(x) = 0, \qquad \mathcal{B}\left(f_0, \frac{df_0}{dx}\right) = 0. \tag{2.7}$$

The auxiliary function $H(r)$ can be written in the form:

$$H(r) = rC_1 + r^2C_2 + r^3C_3 + \ldots \tag{2.8}$$

where C_1, C_2, C_3,... are constants and can be found later.

To obtain the approximate solution, we expand $\kappa(x, r, C_1)$ by Taylor's series about r in the following form,

$$\kappa(x,r,C_1,C_2,\ldots,C_i) = f_0(x) + \sum_{k=1}^{\infty} f_k(x,C_1,C_2,\ldots,C_i)r^k, \quad i = 1,2,\ldots \tag{2.9}$$

Substituting Eq. (2.9) into Eq. (2.1) and Eq. (2.2) and equating the coefficient of like powers of r, we obtain the Zeroth order problem (2.7), the first and second order problems (2.10)-(2.12) respectively. The general governing equations for $f_k(x)$ is given by Eq. (2.12):

$$\mathcal{L}\big(f_1(x)\big) = C_1\mathcal{N}_0\big(f_0(x)\big), \qquad \mathcal{B}\left(f_1, \frac{df_1}{dx}\right) = 0, \tag{2.10}$$

$$\mathcal{L}\big(f_2(x)\big) - \mathcal{L}\big(f_1(x)\big) = C_1\mathcal{N}_0\big(f_0(x)\big) + C_1\big[\mathcal{L}\big(f_1(x)\big) + \mathcal{N}_1\big(f_0(x), f_1(x)\big)\big], \qquad \mathcal{B}\left(f_2, \frac{df_2}{dx}\right) = 0, \tag{2.11}$$

$$\mathcal{L}\big(f_k(x)\big) - \mathcal{L}\big(f_{k-1}(x)\big) = C_k\mathcal{N}_0\big(f_0(x)\big) + \sum_{i=1}^{k-1} C_i\big[\mathcal{L}\big(f_{k-i}(x)\big) + \mathcal{N}_{k-i}\big(f_0(x), f_1(x), \ldots, f_{k-i}(x)\big)\big], \tag{2.12}$$

where $N_{k-1}(f_0(x), f_1(x),\ldots,f_{k-1}(x))$ is the coefficient of r^{k-1} in the series of $N(\kappa(x, r))$ about r.

$$\mathcal{N}\big(\kappa(x,r,C_i)\big) = \mathcal{N}_0\big(f_0(x)\big) + \sum_{k\geq 1}\mathcal{N}_k\big(f_0, f_1, f_2, \ldots, f_k\big)r^k, \quad i = 1,2,3,\ldots \tag{2.13}$$

It should be underscored that the f_k for $k \geq 0$ are governed by the linear equations with linear boundary conditions obtained from original problem and can be solved easily.

It is observed that the convergence of the Eq. (2.9) depends upon C_1, C_2,...,. If it is convergent at $r = 1$, we have:

$$\tilde{f}(x,C_1,C_2,\ldots C_i) = f_0(x) + \sum_{k\geq 1} f_k(x,C_1,C_2,\ldots C_i). \tag{2.14}$$

Substituting Eq. (2.14) into Eq. (2.1), we obtained an expression for residual

$$R(x,C_1,C_2,\ldots C_i) = \begin{bmatrix} \mathcal{L}(\tilde{f}(x,C_1,C_2,\ldots C_i)) + h(x) \\ +\mathcal{N}(\tilde{f}(x,C_1,C_2,\ldots C_i)) \end{bmatrix}. \tag{2.15}$$

If $R(x, C_1, C_2,\ldots,C_i) = 0$ then $\tilde{f}(x,C_1,C_2,\ldots C_i)$ is the exact solution of the problem. Usually it doesn't occur, particularly in nonlinear BVP's.

For finding constants, C_i, i - 1,2,...,m different methods are used like Galerkin's Method, Ritz Method, Least Squares Method and Collocation Method. One can apply the Method of Least Squares as under:

$$J(C_1,C_2,\ldots C_m) = \int_a^b R^2(x,C_1,C_2,C_3\ldots C_m)\,dx, \tag{2.16}$$

where a and b depends on the nature of the given problem.

The auxiliary constants C_i, i = 1,2,...,m can be optimally found from the Eq. (2.17):

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \ldots = \frac{\partial J}{\partial C_m} = 0. \tag{2.17}$$

The mth order approximate solution can be obtained by these constants so obtained. The constants C_i can also be obtained by Collocation method:

$$R(k_1,C_1,C_2,\ldots C_m) = \begin{bmatrix} R(k_2,C_1,C_2,\ldots,C_m) = \ldots \\ = R(k_m,C_1,C_2,\ldots,C_m) = 0, i = 1,2,\ldots,m \end{bmatrix}. \tag{2.18}$$

The convergence of OHAM is directly proportional to the number of optimal constants C_1, C_2,... which is determined by Eqs. (2.16) - (2.17).

It is observed [11-13] that OHAM reduces to HPM and HAM for $H(r) = -r$ and $H(r) = r\hbar$ respectively, where $\hbar$ is taken as ''$\hbar$-curves''.

Application of Oham to the Flow past a Rotating Disk: To demonstrate the effectiveness of OHAM formulation, consider the steady, laminar flow of an incompressible, viscous fluid past a rotating disk having uniform velocity about the z-axis. Taking the cylindrical coordinates (r, θ, z) and velocity vector (u, v, w). The equations of motion in dimensionless form can be written:

$$\begin{aligned} &f'' - hf' - f^2 + g^2 = 0, \\ &g'' - hg' - 2fg = 0, \\ &2f + h' = 0, \end{aligned} \tag{3.1}$$

with boundary conditions

$$f(0) = 0,\ \ g(0) = 1,\ \ h(0) = 0, f(\infty) = 0,\ g(\infty) = 0, \tag{3.2}$$

In above equations prime denote differential of f, g, h with respect to z.

Applying the method discussed in Section 2, leads to the following:

$$\begin{aligned} &(1-r)f'' - H_1(r)\left[f'' - hf' - f^2 + g^2\right] = 0, \\ &(1-r)g'' - H_2(r)\left[g'' - hg' - 2fg\right] = 0, \\ &(1-r)(2f + h') - H_2(r)\left[2f + h'\right] = 0, \end{aligned} \tag{3.3}$$

We consider:

$$\begin{aligned} &f = f_0 + rf_1 + r^2 f_2, \\ &g = g_0 + rg_1 + r^2 g_2, \\ &h = h_0 + rh_1 + r^2 h_2, \\ &H_1(r) = rC_{11} + r^2 C_{12}, \\ &H_2(r) = rC_{21} + r^2 C_{22}, \\ &H_3(r) = rC_{31} + r^2 C_{32}. \end{aligned} \tag{3.4}$$

Zeroth Order System:

$$\begin{aligned} &f_0'' = 0, \\ &g_0'' = 0, \\ &h_0' + 2f = 0, \end{aligned} \tag{3.5}$$

with boundary conditions

$$f_0(0) = 0,\ \ g_0(0) = 1,\ \ h_0(0) = 0,\ f_0(5) = 0,\ \ g_0(5) = 0. \tag{3.6}$$

We obtain the solution:

$$\begin{aligned} &f_0(z) = 0, \\ &g_0(z) = \frac{5-z}{5}, \\ &h_0(z) = 0, \end{aligned} \tag{3.7}$$

First Order System:

$$\begin{aligned} &C_{11}(f_0)^2 - C_{11}(g_0)^2 + C_{11}h_0 f_0' - f_0'' - C_{11}f_0'' + f_1'' = 0, \\ &2C_{21}f_0 g_0 + C_{21}h_0 g_0' - g_0'' - C_{21}g_0'' + g_1'' = 0, \\ &-2f_0 - 2C_{31}f_0 + 2f_1 - h_0' - C_{31}h_0' + h_1' = 0. \end{aligned} \tag{3.8}$$

With boundary conditions

$$f_1(0) = 0,\ \ g_1(0) = 0,\ \ h_1(0) = 0,\ f_1(5) = 0,\ \ g_1(5) = 0.$$

Its solutions are,

$$\begin{aligned} &f_1 = \frac{1}{300}\left(-375z + 150z^2 - 20z^3 + z^4\right)C_{11}, \\ &g_1 = 0, \\ &h_1 = \frac{1}{1500}\left(1875z - 500z^3 + 50z^4 - 2z^5\right)C_{11}. \end{aligned} \tag{3.9}$$

Second Order System:

$$\begin{aligned} &C_{12}f_0^2 + 2C0_{11}f_0 f_1 - C_{12}g_0^2 - 2C_{11}g_0 g_1 + C_{12}h_0 f_0' + C_{11}h_1 f_0' + C_{11}h_0 f_1' - C_{12}f_0'' - f_1'' - C_{11}f_1'' + f_2'' = 0, \\ &2C_{22}f_0 g_0 + 2C_{21}f_1 g_0 + 2C_{21}f_0 g_1 + C_{22}h_0 g_0' + C_{21}h_1 g_0' + C_{21}h_0 g_1' - C_{22}g_0'' - g_1'' - C_{21}g_1'' + g_2'' = 0, \\ &-2C_{32}f_0 - 2f_1 - 2C_{31}f_1 + 2f_2 - C_{32}h_0' - h_1' - C_{31}h_1' + h_2' = 0. \end{aligned} \tag{3.10}$$

along with,

$$f_2(0)=0,\ g_2(0)=0,\ h_2(0)=0,\ f_2(5)=0,\ g_2(5)=0. \tag{3.11}$$

We get its solutions as

$$f_2=\frac{1}{300}\begin{bmatrix}\left(-375z+150z^2-20z^3+z^4\right)C_{11}+\left(375z+150z^2-20z^3+z^4\right)C_{11}{}^2\\ \left(-375z+150z^2-20z^3+z^4\right)C_{12}\end{bmatrix},$$
$$g_2=\frac{1}{630000}\left[\left(-2109375z+262500z^3-65625z^4+8400z^5-560z^6+16z^7\right)C_{11}C_{12}\right],$$
$$h_2=-\frac{1}{1500}\left[\left(-1875z^2+500z^3-50z^4+2z^5\right)\left(C_{11}+C_{11}{}^2+C_{12}\right)\right]. \tag{3.12}$$

From Eqs. (3.7), (3.9) and (3.12), we

$$f(z)=f_0(z)+f_1(z)+f_2(z),$$
$$g(z)=g_0(z)+g_1(z)+g_2(z), \tag{3.13}$$
$$h(z)=h_0(z)+h_1(z)+h_2(z).$$

$$f(z)=\frac{z}{300}\left[\left(-375+150z-20z^2+z^3\right)\left(2C_{11}+C_{11}{}^2+C_{12}\right)\right],$$
$$g(z)=1-\frac{z}{5}+\frac{z}{63000}\left[\left(-2109375+262500z^2-65625z^3+8400z^4-560z^5+16z^6\right)C_{11}C_{21}\right],$$
$$h(z)=-\frac{z^2}{1500}\left[\left(-1875+500z-50z^2+2z^3\right)\left(2C_{11}+C_{11}{}^2+C_{12}\right)\right]. \tag{3.14}$$

For the computation of the constants C_{11}, C_{12} and C_{21} and applying the method of Least square, we get:

$C_{11}=-0.2051989$, $C_{12}=-0.1109909$, $C_{21}=-0.4979561$.

Putting these values in Eq. (3.14), we obtain:

$$f(z)=-0.00085766(-5+z)\left(75z-15z^2+z^3\right),$$
$$g(z)=2.59505\times10^{-6}\begin{bmatrix}(9.08849+z)\left(3.86957-3.47926z+z^2\right)\left(71.3057+7.41298z+z^2\right)\\ \left(153.666+21.9778z+z^2\right)\end{bmatrix},$$
$$h(z)=0.000343067(-8.6106+z)\left(108.877z^2-16.3894z+z^2\right). \tag{3.15}$$

RESULTS AND DISCUSIONS

The OHAM formulation given in Section 2 provides highly accurate solutions for the problems given in Section 3. We have used Mathematica 7 for most of our computational work. In Table 1, OHAM solution of *f*, *g*, *h* are given at different values of *z* and in Table 2, HPM solution of *f*, *g*, *h* are given at different values of *z*. In Table 3, the absolute errors of the values of *f*, *g*, *h* obtained by OHAM with respect to the results obtained by HPM. Fig. 1 and 2 shows the plots for the results obtained by OHAM and HPM for the system of equations (3.1). From Tables, 1-3 and Figs. 1-2 it is clear that OHAM results and HPM results are nearly identical. It has been observed that the absolute errors for *f*, *g*, *h* are decreases by increasing the values of *z*.

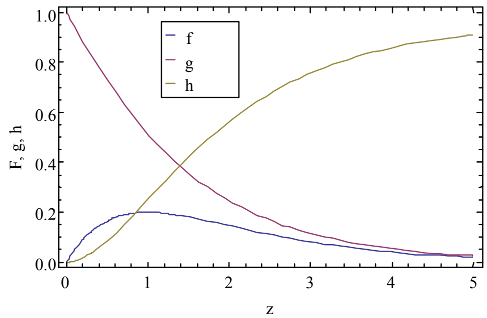

Fig. 1: Variation of velocity components *f*, *g*, *h* with *z*, the dimensionless distance normal to the disk obtained by OHAM

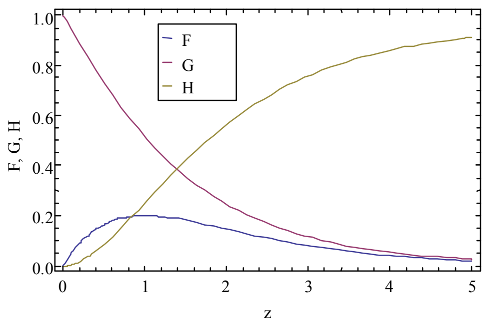

Fig. 2: Variation of velocity components *f*, *g*, *h* with *z*, the dimensionless distance normal to the disk obtained by HPM

Table 1: Result obtained by OHAM for the system of equation (3.1)

z	*f*	*g*	*h*
0	0.0000000	1.000000	0.0000000
0.1	0.0308931	0.945832	0.0031313
0.2	0.0593149	0.891934	0.0121925
0.3	0.0853653	0.838603	0.0266992
0.4	0.1091420	0.786163	0.0461870
0.5	0.1307410	0.734971	0.0702108
0.6	0.1502550	0.685413	0.0983444
0.7	0.1677777	0.637914	0.1301800
0.8	0.1833950	0.592932	0.1653280
0.9	0.1971980	0.550966	0.2034170
1.0	0.2092271	0.512554	0.2440920

Table 2: Result obtained by HPM for the system of equation (3.1)

z	*f*	*g*	*h*
0	0.00000	1.000000	0.000000
0.1	0.0485707	0.943165	0.0043705
0.2	0.0882978	0.887159	0.0164216
0.3	0.1202690	0.832600	0.0347072
0.4	0.1454910	0.779923	0.0579643
0.5	0.164879	0.729420	0.0850973
0.6	0.179264	0.681274	0.1151630
0.7	0.189388	0.635587	0.1473540
0.8	0.195912	0.592394	0.1809880
0.9	0.199419	0.551686	0.2154890
1.0	0.200423	0.513419	0.2503750

Table 3: Absolute errors of the result obtained by OHAM corresponding to the result obtained by HPM for the system of equation (3.1)

z	Abs*f*	Abs*g*	Abs*h*
0	0.0000000	0.000000	0.0000000
0.1	0.0176775	0.0026666	0.0012392
0.2	0.0289828	0.0047749	0.0042290
0.3	0.0349042	0.0060028	0.0080079
0.4	0.0363487	0.0062402	0.0117773
0.5	0.0341384	0.0055072	0.0148865
0.6	0.0216109	0.0041389	0.0168184
0.7	0.0125163	0.0023275	0.0171744
0.8	0.0125163	0.0053864	0.0156601
0.9	0.0022210	0.0007200	0.0120715
1.0	0.0088434	0.0003654	0.0062831

CONCLUSION

In this work, OHAM [7-12] has been proved to be useful tool for solving the system of three differential equations with boundary conditions. The OHAM has been applied to Eq. (3.1) and found simpler in applicability, useful to control convergence and involved less computational. Therefore, OHAM displays its validity and great potential for the solution nonlinear system of differential equations (3.1) BVP's.

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