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An Iterative Method for Solving a Symmetric System of Fuzzy Linear Equations

[1]Majid Hasanzadeh and [2]H. Zareamoghaddam

[1]Torbat-e-Jam Branch, Islamic Azad University, Torbat-e-Jam, Iran
[2]Young Researchers Club, Islamic Azad University, Kashmar Branch, Kashmar, Iran

Abstract: There are several iterative methods to solve fuzzy linear system of equations while the number of proposed methods for solving a crisp linear system of equations is much more. Surely, all of crisp methods will not be applicable for fuzzy problems, however, scientists attempt to develop the number of fuzzy solvers. In this study, one of these techniques based on chebyshev iteration is applied for solving fuzzy symmetric linear systems (FSLS). At the end of this paper, the new algorithm is illustrated by solving some numerical examples.

Key words: *Fuzzy linear system · Iterative methods · Fuzzy number · Linear system of equations*

INTRODUCTION

There are many applications for the solution of linear system of equations. Due to the importance of these solutions for various problems, scientists have paid especial attention to solve such problems by iterative methods as ease and fast as possible. So, it is important to study more about variety of efficient techniques for solving linear system of equations [1-3]. In some cases, at least some of the system's parameters are represented by fuzzy rather than crisp numbers to fit more the problem with its nature that this problem is extracted from. The solution of such a fuzzy linear system is applicable in different areas such as economic, engineering, physics etc. Therefore, it is essential to develop solving methods, which appropriately treat fuzzy linear systems, for finding suitable solutions.

For the first time, the concept of fuzzy numbers and associated arithmetic operations was proposed by Zadeh. Next, a general model for solving a $n \times n$ fuzzy linear system of equations(FLSE), where the coefficient matrix is crisp and the right hand side is a fuzzy vector, was proposed by Friedman *et al* [4, 5]. After that, in literature, various methods have been proposed for solving FLSE [6-9] in which many of them based on some popular solver methods for crisp linear systems [10].

This paper is organized as follows. In section 2, we discuss some basic definitions and results on fuzzy numbers and the fuzzy linear system of equations. The proposed model for solving FSLS by chebeshev algorithm is discussed in section 3. Numerical tests and conclusion are drawn in the next sections.

Basic Concepts and Definitions: Here, some primary definitions and notes, which are required in this study, have been indicated from [4-7].

Definition 1: The r-level set of a fuzzy set $\tilde{u}$ is defined as an ordinary set $[\tilde{u}]_r$ of which the degree of membership function exceeds the level r, i.e.

$$\tilde{u}_r = [\tilde{u}]_r = \left\{ x \in R \,\middle|\, \tilde{u}(x) \geq r \quad , r \in [0,1] \right\} \tag{1}$$

Definition 2: A fuzzy set $\tilde{u}$, defined on the universal set of real number R, is said to be a fuzzy number if its membership function has the following characteristics:

- $\tilde{u}$ is convex i.e.

$$\mu_{\tilde{u}}(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_{\tilde{u}}(x_1), \mu_{\tilde{u}}(x_2)) \forall x_1, x_2 \in R, \forall \lambda \in [0,1] \tag{2}$$

- $\tilde{u}$ is normal i.e. $\exists x_0 \in R$ such that $\mu_{\tilde{u}}(x_0). = 1$
- $\mu_{\tilde{u}}$ is piecewise continuous.

Definition 3: A fuzzy number $\tilde{u}$ in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$, $0 \leq r \leq 1$, that satisfies the following requirement:

Corresponding Author: Majid Hasanzadeh, Torbat-e-Jam Branch, Islamic Azad University, Torbat-e-Jam, Iran.

- $\underline{u}(r)$ is a bounded monotonically increasing left continuous function;
- $\overline{u}(r)$ is a bounded monotonically decreasing left continuous function;
- $\underline{u}(r) \le \overline{u}(r),\ 0 \le r \le 1$.

Definition 4: The addition and scalar multiplication of fuzzy numbers are defined by the extension principle and can be equivalently represented as follows, see Friedman *et al*. [6, 7].

For arbitrary $\tilde{u} = (\underline{u}, \overline{u})$, $\tilde{v} = (\underline{v}, \overline{v})$ and k $\in R$, the addition and the scalar multiplication are defined as follows:

- $\tilde{u} = \tilde{v}$ iff $\underline{u}(r) = \underline{v}(r)$ and $\overline{u}(r) = \overline{v}(r)$,
- $(\tilde{u} \pm \tilde{v})(r) = \left((\underline{u}(r) \pm \underline{v}(r)), (\overline{u}(r) \pm \overline{v}(r))\right)$,
- $k\tilde{u} = \begin{cases} (k\underline{u}, k\overline{u})(r), & k \ge 0, \\ (k\overline{u}, k\underline{u})(r), & k < 0. \end{cases}$

Remark 1: A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha,\ 0 \le r \le 1$.

Definition 5: The triangular fuzzy number $\tilde{u} = (u_1, u_2, u_3)$ is a fuzzy set where the membership function is as

$$\tilde{u}(x) = \begin{cases} \dfrac{x - u_1}{u_2 - u_1}, & u_1 \le x \le u_2, \\ \dfrac{u_3 - x}{u_3 - u_2}, & u_2 \le x \le u_3, \\ 0, & Otherwise; \end{cases} \tag{3}$$

and its parametric form is

$$\tilde{u} = (\underline{u}(r), \overline{u}(r)),\ \underline{u}(r) = (u_2 - u_1)r + u_1,\ \overline{u}(r) = u_3 - (u_3 - u_2)r \tag{4}$$

Definition 6: A triangular fuzzy number $\tilde{u}$ is said to be non-negative fuzzy number if and only if $\tilde{u}(x) = 0, \forall x < 0$.

For solving a $n \times n$ fuzzy linear system

$$A\tilde{x} = \tilde{b} \tag{5}$$

with a crisp square matrix A and a triangular fuzzy vector $\tilde{b}$ different iterative methods have been proposed.

The ith row of fuzzy linear system with the solution $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)^T$, $\tilde{x}_i = (\overline{x}_i(r), \underline{x}_i(r))$, $i = 1,\ldots,n$ is as

$$\underline{\sum_{j=1}^{n} a_{i,j} x_j} = \sum_{j=1}^{n} \underline{a_{i,j} x_j} = \underline{b_i},$$

$$\overline{\sum_{j=1}^{n} a_{i,j} x_j} = \sum_{j=1}^{n} \overline{a_{i,j} x_j} = \overline{b_i},\ i = 1,\ldots,n$$

From the above, it is extracted that

$$\underline{a_{i,1}x_1} + \cdots + \underline{a_{i,n}x_n} = \underline{b_i}(r)$$

$$\overline{a_{i,1}x_1} + \cdots + \overline{a_{i,n}x_n} = \overline{b_i}(r)$$

Based on the idea of Friedman *et al*, the system (5) is converted into a $2n \times 2n$ crisp function linear system

$$SX = Y \tag{6}$$

where matrix $S = (s_{ij})$ is obtained as follows:

$$\begin{cases} a_{ij} \ge 0 \ \Rightarrow \ s_{ij} = a_{ij}, \quad s_{i+n,j+n} = a_{ij}, \\ a_{ij} < 0 \ \Rightarrow \ s_{i,j+n} = -a_{ij}, \quad s_{i+n,j} = -a_{ij}, \end{cases} \tag{7}$$

and each s_{ij} which is not determined by (7) is zero, $Y = (\underline{b_1}, \ldots, \underline{b_n}, -\overline{b_1}, \ldots, -\overline{b_n})^T$ and $X = (\underline{x_1}, \ldots, \underline{x_n}, -\overline{x_1}, \ldots, -\overline{x_n})^T$.

The matrix S is determined as a symmetric block matrix $S = \begin{pmatrix} B & C \\ C & B \end{pmatrix}$ where $b_{i,j} = s_{i,j}$ and $c_{i,j} = s_{i+n,j}$.

An approximate solution of (5) that is often used is the least square solution of (6), defined as a vector X which minimizes the Euclidean norm of (Y-SX).

Definition 3.1: Let $x = \{(\underline{x}_j(r), \overline{x}_j(r)), 1 \le j \le n\}$ denote the solution of $SX = Y$. The triangular fuzzy vector $\tilde{U} = \{(\underline{u}_j(r), \overline{u}_j(r)), 1 \le j \le n\}$ defined by $\underline{u}_j(r) = \min\{\underline{x}_j(r), \overline{x}_j(r), \underline{x}_j(1)\}$ and $\overline{u}_j(r) = \max\{\underline{x}_j(r), \overline{x}_j(r), \overline{x}_j(1)\}$ is called the fuzzy solution of $SX = Y$. If $\forall j$: $\underline{u}_j = \underline{x}_j$ and $\overline{u}_j = \overline{x}_j$, U is called a strong fuzzy solution, otherwise U is a weak solution.

Solving Fuzzy Linear Systems: In this section, chebyshev iterative method [3] is applied for solving fuzzy symmetric linear system of equations. To have a better view, the chebyshev algorithm is recalled below from [3] for solving a crisp linear system of equations $SX = Y$.

Chebyshev Algorithm:

Compute $R_0 = Y - SX_0$ for initial guess X_0.
$d = (\lambda_{max} + \lambda_{min})/2$, $c = (\lambda_{max} - \lambda_{min})/2$,
for $i = 1,2,\ldots$
  if $i = 1$
    $p_1 = R_0$, $\alpha_1 = 2/d$.
  else
    $\beta_{i-1} = (c\alpha_{i-1}/2)^2$,
    $\alpha_i = 1/(d - \beta_{i-1})$,
    $p_i = R_{i-1} + \beta_{i-1}p_{i-1}$.
  end
  $X_i = X_{i-1} + \alpha_i p_i$,
  $R_i = Y - SX_i$.
End

In this algorithm, λ_{max} and λ_{min} are the largest and smallest eigenvalues of S in point of absolute value, respectively. This algorithm may use for nonsymmetric square matrices; however, the speed of convergence will be too slow.

Numerical Examples

Example 1: Consider the 3×3 fuzzy linear system

$$4\tilde{x}_1 + 2\tilde{x}_2 - \tilde{x}_3 = (-27 + 7r, -7 - 13r),$$
$$2\tilde{x}_1 + 7\tilde{x}_2 + 3\tilde{x}_3 = (-8 + 12r, 25 - 21r),$$
$$-\tilde{x}_1 + 3\tilde{x}_2 + 10\tilde{x}_3 = (29 + 15r, 61 - 17r).$$

As, the coefficient matrix A is symmetric, the matrices B and C and therefore S are symmetric. So, we have

$$S = \begin{pmatrix} 4 & 2 & 0 & 0 & 0 & 1 \\ 2 & 7 & 6 & 0 & 0 & 0 \\ 0 & 6 & 10 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & 0 & 2 & 7 & 6 \\ 1 & 0 & 0 & 0 & 6 & 10 \end{pmatrix}, \quad Y = \begin{pmatrix} -27+7r \\ 1+15r \\ 26+18r \\ 7+13r \\ -40+24r \\ -47+33r \end{pmatrix}$$

For eigenvalues of matrix S, λ_{max} = 12.3004 and λ_{min} = 2.2708. The exact solution of this system is as

$$\begin{cases} \tilde{x}_1 = (-5 + r, -2 - 2r), \\ \tilde{x}_2 = (-1 + r, 2 - 2r), \\ \tilde{x}_3 = (3 + r, 5 - r). \end{cases}$$

The approximated solution of this linear system has been computed by Chebyshev algorithm with $\|X - X_n\| < 10^{-10}$ after 38 iterates.

Example 2: Now, the following 4×4 fuzzy linear system is considered

$$5\tilde{x}_1 + 2\tilde{x}_2 - \tilde{x}_3 = (11 + 8r, -4 - 8r),$$
$$2\tilde{x}_1 + 4\tilde{x}_2 + \tilde{x}_4 = (11 + 8r, 8 - 5r),$$
$$-\tilde{x}_1 + 2\tilde{x}_3 = (7 + 3r, -3r),$$
$$\tilde{x}_2 + 3\tilde{x}_4 = (10 + 7r, -8 - 2r),$$

As, the coefficient matrix A is symmetric, the matrices B and C and therefore S are symmetric. So, we have

$$S = \begin{pmatrix} 5 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 5 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 4 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 \end{pmatrix}, \quad Y = \begin{pmatrix} 11+8r \\ 11+8r \\ 7+3r \\ 10+7r \\ 4+8r \\ -8+5r \\ 3r \\ -8-2r \end{pmatrix}.$$

For eigenvalues of matrix S, λ_{max} = 6.7913 and λ_{min} = 1.3820. The exact solution of this system is

$$\begin{cases} \tilde{x}_1 = (2 + r, -1 - r), \\ \tilde{x}_2 = (1 + r, 2 - r), \\ \tilde{x}_3 = (3 + r, 1 - r), \\ \tilde{x}_4 = (3 + 2r, 2 + r). \end{cases}$$

The approximated solution of this problem has been computed after 38 iterates.

CONCLUSION

Fuzzy linear system of equations is an applicable problem that its solution is important in many areas. I this paper, the chebyshev iteration method has been applied for finding the solution of fuzzy symmetric linear system of equations. Numerical example shows that this algorithm is practical for solving such problems.

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