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Quintic B-Spline for the Numerical Solution of Fourth-Order Parabolic Partial Differential Equations

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Abstract: The quintic B-spline collocation method is developed to solve the fourth-order parabolic partial differential equation. This kind of problem arises in the field of transverse vibration of the uniform flexible beam. Stability analysis of the method has also been proven. Two examples have been considered to illustrate the efficiency of the method developed. It has been observed that the numerical results efficiently approximate the exact solutions.

Key words: Quintic B-spline . collocation method . stability analysis . fourth-order parabolic equation . convergence analysis

INTRODUCTION

The problem of undamped transverse vibrations of a flexible straight beam whose supports do not contribute to the strain energy of the system, is considered. The problem is represented in the following fourth-order parabolic partial differential equation

$$\frac{\partial^2 u}{\partial t^2}+\mu\frac{\partial^4 u}{\partial x^4}=f(x,t),\ x\in[0,1],\ \mu>0,\ t>0 \qquad (1)$$

subject to the initial conditions

$$\begin{aligned} u(x,0)&=g_0(x)\\ u_t(x,0)&=g_1(x) \end{aligned} \qquad (2)$$

with the boundary conditions

$$\begin{aligned} u(0,t)&=f_0(t), & u(1,t)&=f_1(t)\\ u_{xx}(0,t)&=p_0(t), & u_{xx}(1,t)&=p_1(t) \end{aligned} \qquad (3)$$

where μ is the ratio of flexural rigidity of the beam to its mass per unit length, u is the transverse displacement of the beam, t and x are the time and distance variables respectively. f(x, t) is the dynamic driving force per unit mass and $g_0(x)$, $g_1(x)$, $f_0(t)$, $f_1(t)$, $p_0(t)$ and $p_1(t)$ are continuous functions.

Numerical solution of Eq. (1.1) based on finite difference method after decomposition into a system of second order equation have been proposed by Collatz [1], Conte [2], Crandall [3], Richtmyer and Morton [4], Evans [5] and Todd [6]. Fairweather and Gourlay [7] derived an explicit and implicit schemes which are based on the semi-explicit method of Lees [8] and high accuracy method of Douglas [9] respectively. Aziz *et al*. [10] developed a three time level scheme based of parametric quintic spline and stability analysis has been carried out.

Caglar and Caglar [11] have proposed a fifth-degree B-spline to solve fourth order parabolic partial differential equation and have shown that their scheme produces accurate results. Evans and Yousif [12] have proposed AGE method for solving the fourth order parabolic equation. Sextic spline solution for fourth order parabolic partial differential equation has been proposed by Khan *et al*. [13].

The paper is organized into six sections. A finite difference approximation to discretize the Eq. (1.1) in time derivative is discussed in Section 2. In section 3, quintic B-spline collocation method to solve fourth-order parabolic partial differential equation is discussed. Section 4, presents the way to obtain the initial state which is required to start our scheme. Stability analysis of the proposed method is discussed in section 5. Numerical results are presented in section 6, while the conclusion is presented in section 7.

DISCRETIZATION IN TIME

Consider a uniform mesh Δ with the grid points λ_{ij} to discretize the region Ω = [0,1]×[0, T). Each λ_{ij} is the vertices of the grid point (x_i, t_j) where,

$$x_i = ih, i = 0,1,2,\ldots,N,\ t_j = jk, j = 0,1,2,\ldots$$

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h and k are the mesh sizes in the space and time directions respectively.

The problem is discretized in time variable using the finite difference approximation with step size k

$$\frac{\partial^2 u^n}{\partial t^2} \cong \frac{\delta_t^2 u^n}{k^2(1+\sigma\delta_t^2)} + O(k^4) \qquad (4)$$

where, σ is a parameter such that the finite difference approximation to the time derivative is $O(k^2)$ for arbitrary σ and $O(k^4)$ for σ = 1/2 and σ = 1/4 and σ = 1/6 the finite difference approximations reduce to parametric cubic and cubic spline approximations, respectively.

$$\delta_t^2 u^n = u^{n+1} - 2u^n + u^{n-1}, \; u^n = u(x,t_n)$$

and

$$u^0 = u(x,0) = g_o(x)$$

Substituting Eq. (4) in Eq. (1.1) and discretizing in time variable, we have

$$\frac{\delta_t^2 u^n}{k^2(1+\sigma\delta_t^2)} + \mu u^n_{xxxx} = f(x,t_n) \qquad (5)$$

Thus, it can be rewritten as

$$\begin{aligned} u^{n+1} + \mu\sigma k^2 u^{n+1}_{xxxx} = {} & 2u^n - u^{n-1} - \mu k^2 u^n_{xxxx} + 2\mu\sigma k^2 u^n_{xxxx} \\ & - \mu\sigma k^2 u^{n-1}_{xxxx} + k^2 f(x,t_n) + k^2\sigma(f(x,t_{n+1}) \\ & - 2f(x,t_n) + f(x,t_{n-1})) \end{aligned} \qquad (6)$$

The above equation can be rewritten as

$$\hat{u} + \mu\sigma k^2 \hat{u}_{xxxx} = \hat{\phi}(x) \qquad (7)$$

Where

$$\hat{u} = u^{n+1}, \; \hat{u}_{xxxx} = u^{n+1}_{xxxx}$$

$$\begin{aligned} \hat{\phi}(x) = {} & 2u^n - u^{n-1} - \mu k^2 u^n_{xxxx} + 2\mu\sigma k^2 u^n_{xxxx} \\ & - \mu\sigma k^2 u^{n-1}_{xxxx} + k^2 f^n + k^2\sigma(f^{n+1} - 2f^n + f^{n-1}) \end{aligned}$$

with the boundary conditions

$$\begin{aligned} \hat{u}(0,t) &= f_0(t), & \hat{u}(1,t) &= f_1(t) \\ \hat{u}_{xx}(0,t) &= p_0(t), & \hat{u}_{xx}(1,t) &= p_1(t) \end{aligned} \qquad (8)$$

In each time level, there is an ordinary differential equation in the form of Eq. (7) with the boundary conditions Eq. (8), which is to be solved by quintic B-spline collocation method. The proposed scheme Eq. (7) is a three level scheme. In order to apply the proposed scheme, it is necessary to have the values of u at the nodal points at the zeroth and first level times. To compute u^1, initial conditions

$$u(x,0) = g_0(x)$$

and

$$u_t(x,0) = g_1(x)$$

are used. Since

$$u^0 = u(x,0) = g_0(x)$$

being the value of u at the zeroth level time, therefore using Taylor series for u at t = k, u^1 is computed as

$$u^1 = u^0 + ku^0_t + \frac{k^2}{2!}u^0_{tt} + \frac{k^3}{3!}u^0_{ttt} + \frac{k^4}{4!}u^0_{tttt} + O(k^5) \qquad (9)$$

u^0 and u^0_t are known from initial conditions. This implies that all successive tangential derivatives are known at t = 0, which further shows that $u, u_x, u_{xx}, \ldots, u_t, u_{tx}, u_{txx}, \ldots$ are known at t = 0.

QUINTIC B-SPLINE COLLOCATION METHOD

The interval [0, 1] of domain has been subdivided as

$$0 = x_0 < x_1 < x_2 < \ldots < x_N = 1$$

To provide the support for the quintic B-spline near the end boundaries, ten additional knots have been introduced as

$$x_{-5} < x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_0$$

and

$$x_N < x_{N+1} < x_{N+2} < x_{N+3} < x_{N+4} < x_{N+5}$$

The basis functions

$$B_i(x), \; i = -2, \ldots, N+2$$

of quintic B-spline are defined as

$$B_i(x) = \frac{1}{h^3}\begin{cases} (x - x_{i-3})^5, & x \in [x_{i-3}, x_{i-2}] \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5, & x \in [x_{i-2}, x_{i-1}] \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5 + 15(x - x_{i-1})^5, & x \in [x_{i-1}, x_i] \\ (x_{i+3} - x)^5 - 6(x_{i+2} - x)^5 + 15(x_{i+1} - x)^5, & x \in [x_i, x_{i+1}] \\ (x_{i+3} - x)^5 - 6(x_{i+2} - x)^5, & x \in [x_{i+1}, x_{i+2}] \\ (x_{i+3} - x)^5, & x \in [x_{i+2}, x_{i+3}] \\ 0 & \text{otherwise} \end{cases}$$

The values of successive derivatives

Table 1: Coefficients of quintic B-splines and its derivative at nodes x_i

	x_{i-3}	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}	else
$B_i(x)$	0	1	26	66	26	1	0	0
$B_i^{(1)}(x)$	0	5/h	50/h	0	-50/h	-5/h	0	0
$B_i^{(2)}(x)$	0	$20/h^2$	$40/h^2$	$-120/h^2$	$40/h^2$	$20/h^2$	0	0
$B_i^{(3)}(x)$	0	$60/h^3$	$-120/h^3$	0	$120/h^3$	$-60/h^3$	0	0
$B_i^{(4)}(x)$	0	$120/h^4$	$-480/h^4$	$720/h^4$	$-480/h^4$	$120/h^4$	0	0

$$B_i^{(r)}(x), i=-2,\ldots,N+2; r=0,1,2,3,4$$

at nodes, are listed in Table 1.

For solving Eq. (7) using collocation method with quintic B-spline, an approximate solution $\hat{S}(x)$ to the exact solution of the problem Eq. (7) is to be found. Let $\hat{S}(x)$ can be written in the following form

$$\hat{S}(x)=\sum_{i=-2}^{N+2}\hat{c}_i B_i(x) \tag{10}$$

where

$$\hat{c}_i=(c_0^{n+1},c_1^{n+1},c_2^{n+1},\ldots,c_N^{n+1})$$

are unknown real coefficients and $B_i(x)$ are quintic B-spline functions. Let, $\hat{S}(x)$ satisfies the boundary conditions

$$\hat{S}(x_0)=f_0(t_n),\hat{S}(x_N)=f_1(t_n),\hat{S}^{(2)}(x_0)=p_0(t_n),\hat{S}^{(2)}(x_N)=p_1(t)$$

and the collocation equations

$$L\hat{S}(x_i)=\hat{\phi}(x_i),\ i=0,1,2,\ldots,N \tag{11}$$

where

$$L\hat{S}=\hat{S}+\mu\sigma k^2\hat{S}_{xxxx}$$

Substituting Eq. (10) into Eq. (11) yields the following equation

$$\begin{aligned}
&(c_{i+2}^{n+1}+26c_{i+1}^{n+1}+66c_i^{n+1}+26c_{i-1}^{n+1}+c_{i-2}^{n+1})+\frac{120\mu\sigma k^2}{h^4}(c_{i+2}^{n+1}-4c_{i+1}^{n+1}+6c_i^{n+1}-4c_{i-1}^{n+1}+c_{i-2}^{n+1})=2(c_{i+2}^{n}+26c_{i+1}^{n}+66c_i^{n}+26c_{i-1}^{n}+c_{i-2}^{n})\\
&-(c_{i+2}^{n-1}+26c_{i+1}^{n-1}+66c_i^{n-1}+26c_{i-1}^{n-1}+c_{i-2}^{n-1})-\frac{120\mu k^2}{h^4}(c_{i+2}^{n}-4c_{i+1}^{n}+6c_i^{n}-4c_{i-1}^{n}+c_{i-2}^{n})+\frac{240\mu\sigma k^2}{h^4}(c_{i+2}^{n}-4c_{i+1}^{n}+6c_i^{n}-4c_{i-1}^{n}+c_{i-2}^{n})\\
&-\frac{120\mu\sigma k^2}{h^4}(c_{i+2}^{n-1}-4c_{i+1}^{n-1}+6c_i^{n-1}-4c_{i-1}^{n-1}+c_{i-2}^{n-1})+k^2f_i^n+k^2\sigma(f_i^{n+1}-2f_i^n+f_i^{n-1})
\end{aligned} \tag{12}$$

Simplifying, the above relation leads to the following system of (N +1) linear equations in (N +5) unknowns $c_{-2}^{n+1}, c_{-1}^{n+1}, c_0^{n+1}, c_1^{n+1},\ldots,c_N^{n+1}, c_{N+1}^{n+1}, c_{N+2}^{n+1}$.

$$\begin{aligned}
&c_{i+2}^{n+1}(1+\frac{120\mu\sigma k^2}{h^4})+c_{i+1}^{n+1}(26-\frac{480\mu\sigma k^2}{h^4})+c_i^{n+1}(66+\frac{720\mu\sigma k^2}{h^4})+c_{i-1}^{n+1}(26-\frac{480\mu\sigma k^2}{h^4})+c_{i-2}^{n+1}(1+\frac{120\mu\sigma k^2}{h^4})\\
&=c_{i+2}^{n}(2-\frac{120\mu k^2}{h^4}+\frac{240\mu\sigma k^2}{h^4})+c_{i+1}^{n}(52+\frac{480\mu k^2}{h^4}-\frac{960\mu\sigma k^2}{h^4})+c_i^{n}(132-\frac{720\mu k^2}{h^4}+\frac{1440\mu\sigma k^2}{h^4})\\
&+c_{i-1}^{n}(52+\frac{480\mu k^2}{h^4}-\frac{960\mu\sigma k^2}{h^4})+c_{i-2}^{n}(2-\frac{120\mu k^2}{h^4}+\frac{240\mu\sigma k^2}{h^4})+c_{i+2}^{n-1}(-1-\frac{120\mu\sigma k^2}{h^4})+c_{i+1}^{n-1}(-26+\frac{480\mu\sigma k^2}{h^4})\\
&+c_i^{n-1}(-66-\frac{720\mu\sigma k^2}{h^4})+c_{i-1}^{n-1}(-26+\frac{480\mu\sigma k^2}{h^4})+c_{i-2}^{n-1}(-1-\frac{120\mu\sigma k^2}{h^4})+k^2f_i^n+k^2\sigma(f_i^{n+1}-2f_i^n+f_i^{n-1}),\qquad 0\le i\le N
\end{aligned} \tag{13}$$

To obtain a unique solution of this system, four additional constraints are required. These constraints are obtained from the boundary conditions. Imposition of the boundary conditions enables us to eliminate the parameters $c_{-2}^{n+1}, c_{-1}^{n+1}, c_{N+1}^{n+1}, c_{N+2}^{n+1}$ from the system (13).

After eliminating $c_{-2}^{n+1}, c_{-1}^{n+1}, c_{N+1}^{n+1}, c_{N+2}^{n+1}$ the system (13) is reduced to a penta-diagonal system of (N+1) linear equations in (N + 1) unknowns. This system can be rewritten in matrix form as

$$AC^{n+1} = h^4(BC^n + DC^{n-1} + F) \qquad (14)$$

or

$$A\hat{C} = h^4\hat{Q} \text{ where } \hat{Q} = BC^n + DC^{n-1} + F$$

while,

$$A = \begin{bmatrix} \frac{7200\mu\sigma k^2}{2} & 0 & 0 & 0 & 0 & & & & & \\ p & q & r & s & 0 & & & & & \\ a & b & c & b & a & & & & & \\ & a & b & c & b & a & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & a & b & c & b & a \\ & & & & 0 & s & r & q & p \\ & & & & 0 & 0 & 0 & 0 & \frac{7200\mu\sigma k^2}{2} \end{bmatrix}$$

where

$$a = (h^4 + 120k^2\mu\sigma), \quad b = (26h^4 - 480k^2\mu\sigma), \quad c = (66h^4 + 720k^2\mu\sigma)$$

$$p = (23h^4 - 840k^2\mu\sigma), \quad q = (65h^4 + 600k^2\mu\sigma), \ r = (26h^4 - 480k^2\mu\sigma), s = (h^4 + 120k^2\mu\sigma)$$

$$B = \begin{bmatrix} \frac{3600\mu k^2(-1+2\sigma)}{h^4} & 0 & 0 & 0 & 0 & & & & & \\ x & y & z & v & 0 & & & & & \\ T & R & Z & R & T & & & & & \\ & T & R & Z & R & T & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & T & R & Z & R & T \\ & & & & 0 & v & z & y & x \\ & & & & 0 & 0 & 0 & 0 & \frac{3600\mu k^2(-1+2\sigma)}{h^4} \end{bmatrix}$$

where

$$x = (46 + \frac{840\mu k^2}{h^4} - \frac{1680\mu\sigma k^2}{h^4}), \ y = (130 - \frac{600\mu k^2}{h^4} + \frac{1200\mu\sigma k^2}{h^4}), z = (52 + \frac{480\mu k^2}{h^4} - \frac{960\mu\sigma k^2}{h^4}),$$

$$v = (2 - \frac{120\mu k^2}{h^4} + \frac{240\mu\sigma k^2}{h^4}), T = (2 - \frac{120\mu k^2}{h^4} + \frac{240\mu\sigma k^2}{h^4}), \ R = (52 + \frac{480\mu k^2}{h^4} - \frac{960\mu\sigma k^2}{h^4}), Z = (132 - \frac{720\mu k^2}{h^4} + \frac{1440\mu\sigma k^2}{h^4})$$

$$D = \begin{bmatrix} \frac{-3600\mu\sigma k^2}{h^4} & 0 & 0 & 0 & 0 & & & & & \\ l & m & n & o & 0 & & & & & \\ L & M & N & M & L & & & & & \\ & L & M & N & M & L & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & L & M & N & M & L \\ & & & & 0 & o & n & m & l \\ & & & & 0 & 0 & 0 & 0 & \frac{-3600\mu\sigma k^2}{h^4} \end{bmatrix}$$

where

$$L=(-1-\frac{120k^2\mu\sigma}{h^4}),\ M=(-26+\frac{480k^2\mu\sigma}{h^4}),\ N=(-66-\frac{720k^2\mu\sigma}{h^4}),l=(-23+\frac{840k^2\mu\sigma}{h^4}),$$

$$m=(-65-\frac{600k^2\mu\sigma}{h^4}),n=(-26+\frac{480k^2\mu\sigma}{h^4}),o=(-1-\frac{120k^2\mu\sigma}{h^4})$$

and

$$F=\begin{bmatrix} k^2f_0^n+k^2\sigma(f_0^{n+1}-2f_0^n+f_0^{n-1})-\frac{15k^2\mu(-4f_0+h^2p_0)}{2h^4} \\ k^2f_1^n+k^2\sigma(f_1^{n+1}-2f_1^n+f_1^{n-1})+\frac{k^2\mu\sigma(-20f_0+h^2p_0)}{4h^4}+\frac{120\mu k^2}{h^4}(\frac{f_0}{24}-\frac{h^2p_0}{480})(-1+\sigma) \\ k^2f_2^n+k^2\sigma(f_2^{n+1}-2f_2^n+f_2^{n-1}) \\ \vdots \\ k^2f_{N-2}^n+k^2\sigma(f_{N-2}^{n+1}-2f_{N-2}^n+f_{N-2}^{n-1}) \\ k^2f_{N-1}^n+k^2\sigma(f_{N-1}^{n+1}-2f_{N-1}^n+f_{N-1}^{n-1})+\frac{k^2\mu\sigma(-20f_1+h^2p_1)}{4h^4}+\frac{120\mu k^2}{h^4}(\frac{f_1}{24}-\frac{h^2p_1}{480})(-1+\sigma) \\ k^2f_N^n+k^2\sigma(f_N^{n+1}-2f_N^n+f_N^{n-1})-\frac{15k^2\mu(-4f_1+h^2p_1)}{2h^4} \end{bmatrix}$$

Here A, B and D are $(N+1)\times(N+1)$ penta-diagonal matrices and F is an $(N + 1)$ column vector.

The time evolution of the approximate solution $\hat{S}$ is determined by the time evolution of the vector C^n. This is found by repeatedly solving the recurrence relationship, once the initial vector C^0 has been computed from the initial conditions. The recurrence relationship is penta-diagonal and so can be solved using the Thomas algorithm.

INITIAL STATE VECTOR

The initial state vector C^0 can be determined from the initial condition $u(x, 0) = g_0(x)$ which gives $(N+1)$ equations in $(N + 5)$ unknowns. For determining these unknowns, the following relations at the knots, are used

$$u_x(x_0,0)=u^{(1)}(x_0),\qquad u_x(x_N,0)=u^{(1)}(x_N)$$
$$u_{xx}(x_0,0)=u^{(2)}(x_0),\qquad u_{xx}(x_N,0)=u^{(2)}(x_N)$$

which give penta-diagonal system of equations written in the following matrix form

$$GC^0=E \tag{15}$$

$$G=\begin{bmatrix} 54 & 60 & 6 & 0 & 0 & & & & \\ \frac{101}{4} & \frac{135}{2} & \frac{105}{4} & 1 & 0 & & & & \\ 1 & 26 & 66 & 26 & 1 & & & & \\ & 1 & 26 & 66 & 26 & 1 & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & 1 & 26 & 66 & 26 & 1 \\ & & & & 0 & 1 & \frac{105}{4} & \frac{135}{2} & \frac{101}{4} \\ & & & & 0 & 1 & 6 & 60 & 54 \end{bmatrix}$$

It is to be noted that Eq. (15) is solved by a variant form of the Thomas algorithm to get the initial values.

STABILITY ANALYSIS

The stability of the proposed method is investigated by Von Neumann method. For investigating stability, considering $f(x, t) = 0$ in Eq. (13), it can be written as

$$\xi^2-2\gamma\xi+1=0$$

where

$$\gamma=1-\frac{120\mu r^2\sin^4\psi}{(240\mu\sigma r^2+16)\sin^4\psi-210\sin^2\psi+120} \tag{16}$$

$\psi=\frac{1}{2}\theta h$, where θ is the variable in the Fourier expansion and $r=\frac{k}{h^2}$. Applying the Routh-Hurwitz criterion to equation (16), the following necessary and sufficient conditions for (13) to be stable are obtained

$$-1\le 1-\frac{120\mu r^2\sin^4\psi}{(240\mu\sigma r^2+16)\sin^4\psi-210\sin^2\psi+120}\le 1 \tag{17}$$

Simplifying, the left inequality is obtained as

$$(16+60\mu r^2(2\sigma-1))\sin^4\psi-210\sin^2\psi+120\ \ge\ 0$$

which shows that the scheme (13) is unconditionally stable if $\sigma\ge\frac{1}{2}$ and conditionally stable if

$$\sigma<\frac{1}{2},\ r\le\frac{2}{\sqrt{15(1-2\sigma)}}$$

NUMERICAL RESULTS

In this section, the proposed method is tested on the following two test problems. The maximum absolute errors and absolute errors at the points $x = 0.1$, $x = 0.2$, $x = 0.3$, $x = 0.4$ and $x = 0.5$ are calculated by the proposed method.

The numerical results are compared with Evans and Yousif [12], Aziz *et al*. [10] and Caglar and Caglar [11]. Following Table 2, 3, 4 and 6 show that the proposed method gives better results.

Example 1: Following is the fourth order parabolic partial differential equation

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^4 u}{\partial x^4}=(\pi^4-1)\sin\pi x\cos t, x\in[0,1],\ \mu=1,\ t>0$$

subject to the initial conditions

$$u(x,0)=\sin\pi x,$$
$$u_t(x,0)=0 \qquad x\in[0,1]$$

with the boundary conditions

$$u(0,t)=u(1,t)=u_{xx}(0,t)=u_{xx}(1,t)=0$$

The exact solution of the problem is

$$u(x,t)=\sin\pi x\cos t$$

This problem is solved with $h = 0.05$, $k = 0.005$, giving $r = 2$ and with $h = 0.05$, $k = 0.00125$, giving $r = 0.5$, $\sigma<1/4$.

The absolute errors at points $x = 0.1$, $x = 0.2$, $x = 0.3$ $x = 0.4$, $x = 0.5$, computed by the proposed method are compared with Evans [12] and Aziz *et al*. [10] for 10 and 16 time steps, are presented in Table 2.

The absolute errors for $h = 0.05$ and $r = 0.5$ at the midpoint $x = 0.5$ are compared with Evans [12] and Aziz *et al*. [10] for 32, 48 and 64 time steps, are presented in Table 3.

The absolute errors for $h = 0.05$ and $r = 2.0$ at the midpoint $x = 0.5$ are compared with Evans [12] and Aziz *et al*. [10] for 25, 75 and 100 time steps, are presented in Table 4.

From Table 2-4, it is evident that the proposed method is superior to Evans [12] and Aziz *et al*. [10]. Moreover the same problem is solved with different values of r and the computations are carried out for different time steps. The absolute errors at $x = 0.5$ for $r = \sqrt{\frac{1}{6}}$ and $r = \sqrt{\frac{1}{84}}$ are given in Table 5 for $h = 0.05$. The maximum absolute errors for different values of n and for a fixed value of $k = 0.005$ are compared with Caglar and Caglar [11], are given in Table 6.

Table 2: Absolute errors for h = 0.05 at points x = 0.1, x = 0.2, x = 0.3, x = 0.4, x = 0.5

	r	Time steps	x = 0.1	x = 0.2	x = 0.3	x = 0.4	x = 0.5
Proposed method $\sigma<1/4$	2.0	10	1.23×10^{-4}	2.35×10^{-4}	3.24×10^{-4}	3.81×10^{-4}	4.00×10^{-4}
	0.5	16	1.70×10^{-5}	3.23×10^{-5}	4.45×10^{-5}	5.23×10^{-5}	5.50×10^{-5}
Evans [12]	2.0	10	2.2×10^{-4}	4.1×10^{-4}	5.4×10^{-4}	6.2×10^{-4}	6.5×10^{-4}
	0.5	16	2.5×10^{-5}	4.7×10^{-5}	6.6×10^{-5}	7.8×10^{-5}	8.2×10^{-5}
Aziz *et al.* [10] (0,0,1) $\sigma<1/4$	2.0	10	1.5×10^{-4}	2.8×10^{-4}	3.7×10^{-4}	4.2×10^{-4}	4.4×10^{-4}
	0.5	16	3.2×10^{-5}	5.1×10^{-5}	6.2×10^{-5}	6.9×10^{-5}	7.2×10^{-5}

Table 3: Absolute errors at midpoints, x = 0.5, for h = 0.05 and r = 0.5

	32 Time steps	48 Time steps	64 Time steps
Proposed method $\sigma<1/4$	1.889×10^{-4}	3.9723×10^{-4}	6.7281×10^{-4}
Evans [12]	3.1×10^{-4}	6.9×10^{-4}	1.2×10^{-3}
Aziz *et al.* [10] (0,0,1) $\sigma<1/4$	3.0×10^{-4}	7.0×10^{-4}	1.2×10^{-3}

Table 4: Absolute errors at midpoints, x = 0.5, for h = 0.05 and r = 2.0

	25 Time steps	65 Time steps	100 Time steps
Proposed method $\sigma<1/4$	1.8155×10^{-3}	5.6442×10^{-3}	3.7583×10^{-3}
Evans [12]	3.3×10^{-3}	4.1×10^{-3}	3.9×10^{-3}
Aziz *et al.* [10] (0,0,1) $\sigma<1/4$	2.7×10^{-3}	7.8×10^{-3}	3.0×10^{-3}

Table 5: Absolute errors at midpoints, x = 0.5, for h = 0.05

	r	10 time steps	20 time steps	30 time steps
Proposed Method σ<1/4	$\sqrt{\frac{1}{6}}$	1.603×10^{-5}	5.2041×10^{-5}	1.0777×10^{-4}
	$\sqrt{\frac{1}{84}}$	1.1686×10^{-6}	3.7986×10^{-6}	7.8889×10^{-6}

Table 6: Comparison of proposed method with Caglar and Caglar [11] in maximum absolute errors for example 1

N	K	Proposed Method	Caglar and Caglar [11]
121	0.005	1.2669×10^{-5}	9.3252839×10^{-5}
191	0.005	1.0393×10^{-6}	1.0624582×10^{-6}

Table 7: Maximum absolute errors with h = 0.05 for example 2

	r	10 time steps	50 time steps	100 time steps
Proposed method σ<1/4	$\sqrt{\frac{1}{6}}$	9.94×10^{-4}	7.725×10^{-4}	9.7375×10^{-5}
	$\sqrt{\frac{1}{84}}$	2.9946×10^{-4}	2.792×10^{-4}	2.133×10^{-4}

Example 2: Following is the fourth order parabolic partial differential equation

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^4 u}{\partial x^4}=(\pi^4+1)e^t\sin\pi x,\ \ x\in[0,1],\ \ \mu=1,\ \ t>0$$

subject to the initial conditions

$$u(x,0)=\sin\pi x,$$
$$u_t(x,0)=\sin\pi x \qquad x\in[0,1]$$

with the boundary conditions

$$u(0,t)=u(1,t)=u_{xx}(0,t)=u_{xx}(1,t)=0$$

The exact solution of the problem is

$$u(x,t)=e^t\sin\pi x$$

The maximum absolute errors for

$$r=\sqrt{\frac{1}{6}}\ \text{ and } r=\sqrt{\frac{1}{84}}$$

are given in Table 7 for h = 0.05. The computations are carried out for larger time steps.

CONCLUSION

Quintic B-spline collocation method is developed for the approximate solution of fourth-order parabolic partial differential equations. The numerical solution is obtained using new three time level scheme based on a quintic B-spline for space discretization and finite difference technique for time discretization. The proposed method is shown to be unconditionally stable for σ≥1/2 and conditionally stable for σ<1/2 and

$$r\le\frac{2}{\sqrt{15(1-2\sigma)}}$$

It is observed from the tables that the proposed method gives better results on comparison with [10-12].

REFERENCES

1. Collatz, L., 1973. Hermitian methods for initial value problems in partial differential equations. Miller, J.J.H. (Ed.). Topics in Numerical Analysis, Academic Press, New York, pp: 41-61.
2. Conte, S.D., 1957. A stable implicit finite difference approximation to a fourth order parabolic equation. J. Assoc. Comput. Mech., 4: 18-23.
3. Crandall, S.H., 1954. Numerical treatment of a fourth order partial differentail equations. J. Assoc. Comput. Mech., 1: 111-118.
4. Richtmyer, R.D. and K.W. Morton, 1967. Difference methods for initial value problems. 2nd Edn., John Wiley & Sons.
5. Evans, D.J., 1965. A stable explicit method for the finite difference solution of a fourth order parabolic partial differential equation. Comput. J., 8: 280-287.

6. Todd, J., 1956. A direct approach to the proble of stability in the numerical solution of partial differential equations. Commun. Pure Appl. Math., 9: 597-612.
7. Fairweather, G. and A.R. Gourlay, 1967. Some stable difference approximations to a fourth order parabolic partial differential equation. Math. Comput., 21: 1-11.
8. Lees, M., 1961. Alternating direction and semi explicit difference methods for parabolic partial differential equations. Numer. Math., 3: 398-412.
9. Douglas, J., 1956. The solution of the diffusion equation by a high order correct difference equation. J. Math. Phys., 35: 145-151.
10. Tariq Aziz, Arshad Khan and Jalil Rashidinia, 2005. Spline methods for the solution of fourth order parabolic partial differential equations. Appl. Math. Comput., 167: 153-166.
11. Caglar, H. and N. Caglar, 2008. Fifth-degree B-spline solution for a fourth-order parabolic partial differential equations. Appl. Math. Comput., 201: 597-603.
12. Evans, D.J. and W.S. Yousif, 1991. A note on solving the fourth order parabolic equation by the AGE method. Int. J. Comput. Math., 40: 93-97.
13. Arshad Khan, Islam Khan and Tariq Aziz, 2005. Sextic spline solution for solving fourth-order parabolic partial differential equation. Int. J. Comput. Math., 82: 871-879.