World Applied Sciences Journal 22 (2): 264-269, 2013
ISSN 1818-4952

DOI: 10.5829/idosi.wasj.2013.22.02.309

The Solution of Weakly Nonlinear Oscillatory Problems with No Damping Using MAPLE

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Abstract: Weakly nonlinear oscillatory problems in engineering dynamics are frequently presented by nonlinear ordinary differential equations. Very few of such equations have exact solutions, thus the need for approximation techniques. In this study, the classical perturbation method, namely the Lindstedt-Poincare method (L-P method) is discussed. In most work, the L-P method is used to obtain uniformly valid approximate solutions to second order only, as perturbation procedure usually involves cumbersome algebraic manipulations and calculations which are time-consuming and prone to errors. Therefore, in this paper we manage to get the solution to fifth order by using our own built in Maple package based on Maple 14 and we include the graphical presentations of the results.

Key words: Weakly nonlinear oscillator · Nonlinear ODE · perturbation method · Lindstedt-Poincare method

INTRODUCTION

The Lindsted-Poincare method was introduced by Swedish Mathematician and astronomer Lindstedt in 1882 and French Mathematician and theoretical Physicist Poincare in 1886 [1-4]. It involves a single time variable or scale and manages to obtain uniformly valid approximate solutions to nonlinear problems. In this paper, firstly we consider a weakly nonlinear oscillator with no damping described as,

$$\frac{d^2y}{dt^2}+\omega_0{}^2 y=\varepsilon F\left(y,\frac{dy}{dt}\right),\qquad 0<\varepsilon\ \ 1 \tag{1}$$

where is a small positive parameter, often called a perturbation quantity and ω_0 (we take, $\omega_0^2 = 1$) is an angular frequency, which is often referred to as the natural frequency. In this case, the system is free of damping whereby, there is no resisting force, such as air resistance. The periodic solution to the equation (1) is assumed to be of the form,

$$y(t,\varepsilon)=\sum_{m=0}\varepsilon^m y_m(t)+O\left(\varepsilon^{m+1}\right), \tag{2}$$

known as asymptotic expansion.

In 1987, Rand and Armbruster [5] used the computer algebra system MACYSMA in a number of perturbation techniques for systems of coupled oscillators and bifurcation theory. In 1996, Heck [6] and in 1999 Chin and Nayfeh [7] had implemented L-P method using Maple V Release 4, while in 2001, Lopez [8] had implemented L-P method using Maple 8. In all the papers mentioned using the method, only second order approximations were obtained at most. In the next section we will consider the Duffing equation to illustrate the L-P method.

The Duffing Equation: Duffing equation is the most common example of weakly nonlinear oscillatory system with no damping where $\omega_0^2 = 1$ and $dy/dt = 0$. [9] also have chosen the same example and solve it using homotopy perturbation method coupled with Laplace transform and Padé approximants. Let's consider the initial value problem for the equation,

$$\frac{d^2y}{dt^2}+y=-\varepsilon y^3, \tag{3}$$

with condition, $0 < t < \infty$ and $0 < \varepsilon \le 1$. The initial conditions for the above problem,

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$$y(0) = \alpha, y'(0) = 0 \tag{4}$$

and α is a constant. We take initial conditions (4a,b) for the reason that these initial conditions are sufficiently general to cover all physical systems of interest [10]. Nayfeh [11] and Mickens [12] had solved the Duffing equation with initial conditions (4a,b) using L-P method in second order expansion. According to the standard L-P method, a new variable;

$$\tau = \omega t, \tag{5}$$

is introduced, where ω is the frequency of the system that depends on, that is ω = ω( ). Equation (3) then becomes

$$\omega^2 y' + y = -\varepsilon y^3, \;\; \tau > 0, \tag{6}$$

$$y(0) = \alpha, \quad y'(0) = 0 \tag{7a, b}$$

We seek approximate solutions for y and ω in the form of power series in ε as follows,

$$y(\tau, \omega) = y_0(\tau) + \varepsilon y_1(\tau) + ... + \varepsilon^n y_n(\tau) + ... \tag{8}$$

$$\omega(\varepsilon) = 1 + \varepsilon\omega_1 + ... + \varepsilon^n \omega_n + ... \tag{9}$$

where ω_0 has been chosen to be unity, $\omega(0) = \omega_0 = 1$. Substituting equation (8) and (9) into equation (6) and (7a,b) and equating of like power of ε yields the following system of differential equations for successive approximations:

$$O(1): \quad y_0'' + y_0 = 0, \tag{10}$$

$$y_0'(0) = 0, \quad y_0(0) = \alpha \tag{10b,c}$$

$$O(\varepsilon): \quad y_1'' + y_1 = -2\omega_1 y_0'' - y_0{}^3, \tag{11a}$$

$$y_1(0) = y_1'(0) = 0 \tag{11b,c}$$

$$O\left(\varepsilon^2\right): \;\; y_2'' + y_2 = -2\omega_1 y_1'' - \left(\omega_1{}^2 + 2\omega_2\right) y_0'' - 3y_0{}^2 y_1 \tag{12a}$$

$$y_2(0) = y_2'(0) = 0. \tag{12b,c}$$

Equation (10a) with initial conditions (10b,c) has a periodic solution $y_0(\tau) = \alpha\cos\tau$. Substituting $y_0(\tau) = \alpha\cos\tau$ into equation (11a) and using $\cos^3\theta = ¼(\cos 3\theta + 3\cos)$ we obtain the simplified form as,

$$y'' + y_1 = 2\omega_1 \,\alpha \cos \tau - \alpha^3 \cos^3 \tau \tag{13}$$

The particular solution of (13) is;

$$y_1 = \left(\omega_1\alpha - \frac{3}{8}\alpha^3\right)\tau\sin\tau + \frac{\alpha^3}{32}\cos 3\tau \tag{14}$$

Then, the general solution of (13) is;

$$\begin{aligned} y_1(\tau) &= A\cos\tau + B\sin\tau \\ &+ \left(\omega_1\alpha - \frac{3}{8}\alpha^3\right)\tau\sin\tau + \frac{\alpha^3}{32}\cos 3\tau \end{aligned} \tag{15}$$

Using the initial conditions (11b,c), we obtain the constants A= -1/32 and B = 0. Thus the solution of (15) is,

$$\begin{aligned} y_1(\tau) &= \left(\omega_1\alpha - \frac{3}{8}\alpha^3\right)\tau\sin\tau \\ &+ \frac{\alpha^3}{32}\left(\cos 3\tau - \cos\tau\right) \end{aligned} \tag{16}$$

Note that the solution of y_1 contains a mixed-secular term, which makes the expansion nonuniform. For a uniform expansion, we cannot permit such secular terms to appear in y_1, y_2, y_3, … . The secular term can be eliminated by choosing,

$$\omega_1\alpha - \frac{3\alpha^3}{8} = 0 \;\; \rightarrow \;\; \omega_1 = \frac{3\alpha^2}{8} \tag{17}$$

Thus, without the secular term, the equation (16) becomes,

$$y_1(\tau) = \frac{\alpha^3}{32}\left(\cos 3\tau - \cos\tau\right) \tag{18}$$

According to Nayfeh [13], to determine the condition (17), we do not need to determine the particular solution first as done above. Instead, we only need to inspect the inhomogeneous terms in (13) governing y_1 and choose the coefficient of cos τ to be zero.

Substitute $y_0(\tau) = \alpha \cos \tau$ and equation (18) into equation (8), we obtain the first order perturbation solution of (3) is,

$$y(\tau,\varepsilon) = \alpha\cos\tau + \frac{\varepsilon\alpha^3}{32}\left(\cos 3\tau - \cos\tau\right) + \cdots \tag{19}$$

where $\tau = \omega t$ and w( ) = 1 + ($3\alpha^2/8$) + … .

Next, we solve the initial value problem (12a,b,c) to obtain the solution of second order expansion. The general solution to the resulting differential equation for y_2(t) is,

$$y_2(\tau) = \frac{\alpha^5}{1024}\left(23\cos\tau - 24\cos 3\tau + \cos 5\tau\right) \tag{20}$$

Therefore, to the second order expansion, the solution to equation (3) is,

$$y(\tau,\varepsilon) = \alpha\cos\tau + \frac{\varepsilon\alpha^3}{32}\left(\cos 3\tau - \cos\tau\right) + \frac{\varepsilon^2\alpha^5}{1024}\begin{pmatrix}23\cos\tau - 24\cos 3\tau \\ -\cos\tau\end{pmatrix} + \cdots \tag{21}$$

where τ = ωt and w( ) = 1 + (3α²/8) - ²(21α⁴/256) + … . However according to Nayfeh [13], equation (21) can be written as,

$$y(t,\varepsilon) = \alpha\left(\cos\left(t + \frac{3\varepsilon t\alpha^2}{8} - \frac{21\varepsilon t\alpha}{256}\right)\right) + \frac{\varepsilon\alpha^3}{1024}\begin{pmatrix}\cos 3\begin{pmatrix} t + \dfrac{3\varepsilon t\alpha^2}{8} \\ -\dfrac{21\varepsilon^2 t\alpha^4}{256}\end{pmatrix} \\ -\cos\begin{pmatrix} t + \dfrac{3\varepsilon t\alpha^2}{8} \\ -\dfrac{21\varepsilon^2 t\alpha^4}{256}\end{pmatrix}\end{pmatrix} + O\left(\varepsilon^2\right) \tag{22}$$

which is valid when $t = O(^{-1})$. This is the solution of the Duffing equation in second order expansion. Next, we are going to illustrate the solution of the Duffing equation by L-P method using Maple software up to fifth order expansion.

The Duffing Equation with Maple: Anyone who uses perturbation methods is struck almost immediately by the amount of algebraic manipulation. As an alternative, we can use a symbolic language like Maple. This mathematical software has a tremendous potential for constructing asymptotic expansions [2].

In this section, we will illustrate the initial value problem of Duffing equation (3) with initial conditions (4a,b) by using L-P method with Maple. All commands that we use for solving initial value problems are under the DEtools package in Maple based on Lopez's work [8]. In this work we manage to obtain all the solutions in the fifth order expansions and good agreement with Runge-Kutta Fifth Fourth order (rkf45) numerical solution.

The solution of the Duffing equation in fifth order expansion is obtained as follows,

$$\begin{aligned} y(\tau,\varepsilon) = {} & \alpha\cos\tau + \begin{pmatrix} -\dfrac{\alpha^3}{32}\cos\tau \\ +\dfrac{\alpha^3}{32}\cos 3\tau \end{pmatrix}\varepsilon \\ & + \begin{pmatrix} \dfrac{23\alpha^5}{1024}\cos\tau - \dfrac{3\alpha^5}{128}\cos 3\tau \\ +\dfrac{\alpha^5}{1024}\cos 5\tau \end{pmatrix}\varepsilon^2 \\ & + \begin{pmatrix} -\dfrac{547\alpha^7}{32768}\cos\tau + \dfrac{297\alpha^7}{16384}\cos 3\tau \\ -\dfrac{3\alpha^7}{2048}\cos 5\tau + \dfrac{\alpha^7}{32768}\cos 7\tau \end{pmatrix}\varepsilon^3 \\ & + \begin{pmatrix} \dfrac{6713\alpha^9}{524288}\cos\tau - \dfrac{15121\alpha^9}{1048576}\cos 3\tau \\ +\dfrac{883\alpha^9}{524288}\cos 5\tau - \dfrac{9\alpha^9}{131072}\cos 7\tau \\ +\dfrac{\alpha^9}{1048576}\cos 9\tau \end{pmatrix}\varepsilon^4 \\ & + O\left(\varepsilon^5\right) \end{aligned} \tag{23}$$

where $\tau = t\begin{pmatrix} 1 + 3\alpha^2\varepsilon/8 - 21\alpha^4\varepsilon^2/256 + 81\alpha^6\varepsilon^3/2048 \\ -6549\alpha^8\varepsilon^4/262144 + 37737\alpha^{10}\varepsilon^5/2097152 \end{pmatrix}$

We also present the numerical values of the approximate solutions for (23) together with the graphs for various values of ε and several expansions in different order. Table 1 and Fig. 1 show that the solutions do not vary much for values of ε which are less than 0.05. However, for ε = {-0.1, 0.1, 0.2} show bigger deviation compared to the other values of ε. Note that, ε < 0 and ε > 0 represent pendulum type equations with a hard spring and a soft spring respectively [11].

Table 2(a) and Figure 2(a) shows that at the interval time $0 \le t \le 5$ does not give any discrepancies for different order of expansion. While Table 2(b) and Figure 2(b) show that the higher order expansion gives remarkable accuracy. When $t = 99$, the relative error of a new result is the smallest compared to the Mickens and Lopez's result. From a quantitative point of view, we can conclude that the higher order approximation is important in giving the accurate result.

Other Examples: In this section we will consider the other examples of weakly nonlinear oscillator with no damping. The following equation is a pendulum type equation,

Table 1: The approximate solutions of Duffing equation in fifth order expansion for various values of ϵ tabulated in the interval 0 b t b 11

	ε					
t	0.005	0.01	0.025	0.05	0.1	0.2
1	1.066	1.052	1.009	0.941	0.811	0.577
2	-0.858	-0.883	-0.955	-1.068	-1.27	-1.603
3	-1.986	-1.991	-1.999	-1.994	-1.919	-1.561
4	-1.260	-1.211	-1.063	-0.809	-0.3	0.638
5	0.637	0.705	0.9	1.190	1.644	1.999
6	1.943	1.962	1.996	1.977	1.686	0.514
7	1.435	1.360	1.115	0.673	-0.228	-1.643
8	-0.408	-0.521	-0.843	-1.306	-1.896	-1.517
9	-1.873	-1.915	-1.991	-1.948	-1.328	0.7
10	-1.591	-1.495	-1.167	-0.533	0.743	1.995
11	0.172	0.333	0.786	1.414	1.998	0.452

Table 2(a): The approximate solutions of the Duffing equation at different orders of expansions tabulated in the interval 0 b t b 5

t	rkf45 Lopez (2001)	$O(\varepsilon^2)$ Mickens (1996)	Error	$O(\varepsilon^3)$ Lopez (2001)	Error	$O(\varepsilon^5)$ New results	Error
0	2	2	0.00	2	0.00	2	0.00
1	0.811	0.817	0.006	0.809	0.002	0.811	0.00
2	-1.27	-1.2663	0.004	-1.2708	0.001	-1.27	0.00
3	-1.919	-1.923	0.004	-1.918	0.001	-1.919	0.00
4	-0.301	-0.317	0.016	-0.296	0.005	-0.3	0.001
5	1.643	1.633	0.01	1.646	0.003	1.644	0.001

Table 2(b): The approximate solutions of Duffing equation at different orders of expansions tabulated in the interval 95 b t b 100

t	rkf45Lopez (2001)	$O(\varepsilon^2)$ Mickens (1996)	Error	$O(\varepsilon^3)$ Lopez (2001)	Error	$O(\varepsilon^5)$ New results	Error
95	0.361	0.723	0.362	0.271	0.09	0.354	0.007
96	-1.606	-1.345	0.261	-1.661	0.055	-1.611	0.005
97	-1.721	-1.891	0.17	-1.668	0.053	-1.716	0.005
98	0.166	0.218	0.052	0.259	0.093	0.174	0.008
99	1.873	1.692	0.181	1.906	0.033	1.876	0.003
100	1.375	1.643	0.268	1.301	0.074	1.369	0.006

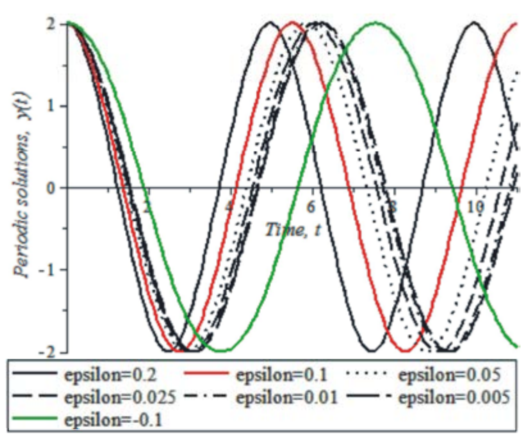

Fig. 1: The approximate solutions of Duffing equation for various values of ε, with initial condition $y(0) = 2, y^{Œ}(0) = 0$ at the interval $0 \le t \le 11$.

$$\frac{d^2y}{dt^2} + y = \varepsilon y^3 \tag{24}$$

Then, we consider the equation,

$$\frac{d^2y}{dt^2} + y = -\varepsilon y^2 \tag{25}$$

Both equations will be solved with initial conditions (4a,b). In the literature, Jordan and Smith [10] had solved the problem in the equation (24) and obtained first order expansion using L-P method. Therefore, with the aid of Maple we obtain fifth order expansion as illustrated in the following Fig. 3(a) and 3(b). Here, we have shown our present solution is in agreement with the numerical solution by the method of Fehlberg fourth-fifth order Runge-Kutta. Jordan and Smith's result in the interval time $0 \le t \le 5$ in Fig. 3(a) and $95 \le t \le 100$ in Fig. 3(b) deviate as time increasing.

Fig. 4(a) and 4(b) show, we present result of the equation (25) and it is in agreement with the numerical solution obtained from the maple package. Both solutions indicate that the higher order solution gives more accurate results of the numerical solution compared to the solution obtained in $O(\varepsilon^2)$.

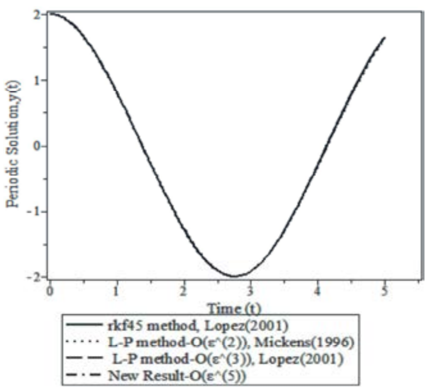

Fig. 2(a): The approximate solutions of Duffing equation at different orders of expansions, with initial conditions $y(0) = 2$, $y'(0) = 0$ and $\varepsilon = 0.1$ in the interval $0 \leq t \leq 5$

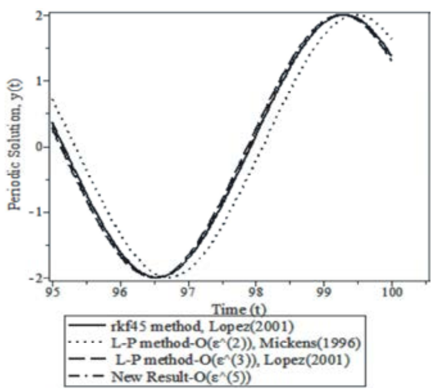

Fig. 2(b): The approximate solutions of Duffing equation at different orders of expansions, with initial conditions $y(0) = 2$, $y'(0) = 0$ and $\varepsilon = 0.1$ at the interval, $95 \leq t \leq 100$

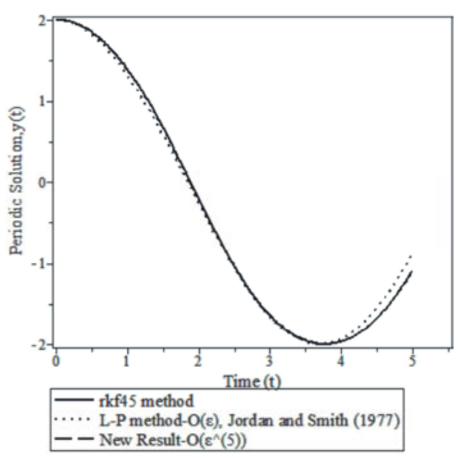

Fig. 3(a): The approximate solutions of equation (24) at different orders of expansions, with initial conditions $y(0) = 2$, $y'(0) = 0$ and $\varepsilon = 0.1$ in the interval $0 \leq t \leq 5$

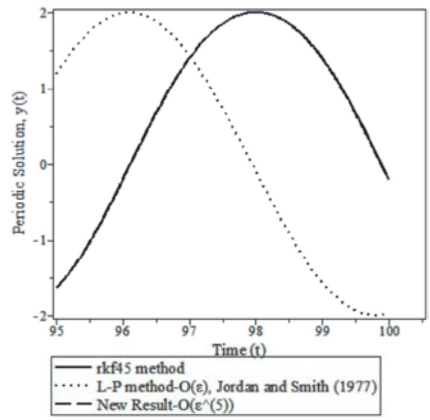

Fig. 3(b): The approximate solutions of equation (24) at different orders of expansions, with initial conditions $y(0) = 2$, $y'(0) = 0$ and $\varepsilon = 0.1$ in the interval $95 \leq t \leq 100$

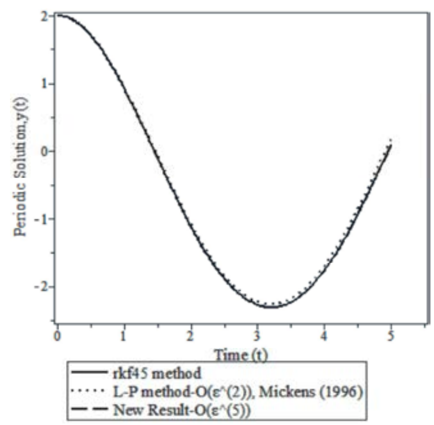

Fig. 4(a): The approximate solutions of equation (25) at different orders of expansions, with initial conditions $y(0) = 2$, $y'(0) = 0$ and $\varepsilon = 0.1$ in the interval $0 \leq t \leq 5$

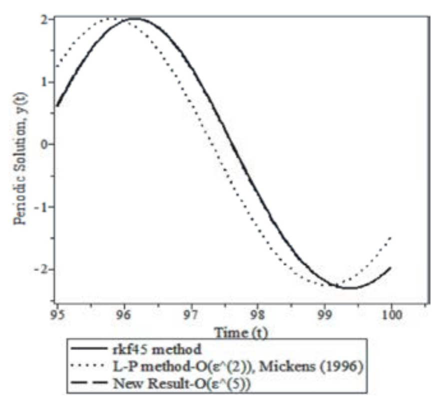

Fig. 4(b): The approximate solutions of equation (25) at different orders of expansions, with initial conditions $y(0) = 2$, $y'(0) = 0$ and $\varepsilon = 0.1$ in the interval $95 \leq t \leq 100$

CONCLUSIONS

In this article, we have presented the approximate solutions of weakly nonlinear oscillator with no damping by considering different initial conditions using L-P method. We implemented L-P method with Maple and manage to get the periodic solutions in higher order expansions and well-fitted with the numerical solution by rkf 45 method. We also have considered the different values of ε and found that the solutions do not vary much for values of ε that's less than 0.025. In future work, we attempt to find the higher order solution for damping oscillatory systems.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge the financial support provided by short term grants, PJP UM grant.

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