

On the Investigation of Quantum Evolution of a Single Photon Wavepacket: A Simulation Approach

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Abstract: Investigation into photon wavepacket is fundamental to the deep understanding of electromagnetic fields as well as light-matter interaction. We studied quantum evolution of a photon wavepacket using numerical simulation. We devised a Hilbert space with sinusoidal basis functions and computed vector elements of the photon wavepacket in the Hilbert space through Fourier transform. We also formed the associated Hamiltonian in the matrix form and derived equations for time evolution of the wavepacket from standard Schrodinger equation. Finally, we performed simulation of the quantum evolution of the wavepacket through repeated application of the Hamiltonian on it. Simulation results showed efficacy of the approach.

Key words: Photon Wavefunction • Photon Wavepacket • Hilbert Space • Quantum Evolution.

INTRODUCTION

Investigation into the nature of light has been a fundamental endeavor among scientists for centuries. Particularly, invention of LASER raised great interests in the research of electromagnetic fields. Non-classical properties of light has been an active area of research since then. Also, need for quantization of light energy, around the beginning of the 20th century, prompted scientists to conceive of a very tiny amount of light energy, called a quantum of light energy – which was later called a ‘photon’.

A photon can be represented by a wavepacket which can be identified with wavefunction in quantum theory [1]. As such, the words, ‘wavepacket’ and ‘wavefunction’ for photons are interchangeably used in this article. Understanding the properties and evolution of this photon wavepacket is fundamental to the investigation into the nature of light.

A large number of researchers from relevant backgrounds have put efforts in the research related to photon wavefunction [2-9]. Among them, Chan *et al.* [2] investigated into photon localization and single-photon wave functions for the photons spontaneously emitted from an excited atom in free space. Baek *et al.* [3] experimentally demonstrated temporal shaping of a single-photon wave packet in the process of spontaneous parametric down conversion. Also, Tamburini and Vicino

[4] examined the limits of photon wavefunction formalism and devised a Dirac-like equation for photon wavefunction in a covariant form.

In this article, we investigate quantum evolution of a photon wavepacket in the complex phase-plane through numerical simulation. After presenting an overview of our approach, we discuss on the mathematical representation of the wavepacket. Then we present a discussion on the formation of the Hilbert space and the associated Hamiltonian matrix, followed by formulation of the equation of the motion for the wavepacket system. Simulation results are finally presented.

Overview: Our simulation approach consists of devising a Hilbert space through selection of a set of basis function, mapping a photon wavepacket into this Hilbert space, forming an appropriate Hamiltonian operator and then applying the Hamiltonian on the wavepacket to obtain its time evolution. A set of sinusoidal functions, $\sin(nx)$, $n = 1, 2, 3, \dots, D$ were adopted as basis functions. These basis functions span an D -dimensional Hilbert space. Photon wavefunction was represented by a vector $|\psi\rangle$ in this Hilbert space. Also, an $D \times D$ Hamiltonian, \hat{H} , was devised in the Hilbert space, which was applied on $|\psi\rangle$ to obtain its time evolution through Schrodinger equation [10]. We performed numerical simulation of time evolution of photon wavepacket and observed its phase evolution.

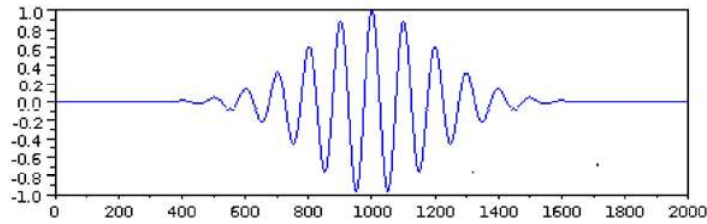


Fig. 1: A position space photon wavepacket.

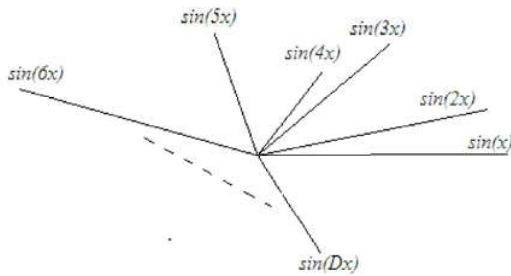


Fig. 2: The Hilbert space.

Photon Wavepacket: In order to simplify our investigation, we considered the photon wavepacket *et al.* along one spatial dimension, namely, x -axis. Also, we assumed that photon spatial movement is confined to the y - z plane and therefore along x -axis the photon wavepacket is stationary. If the x -axis is discretized into N points, $0, 1, 2, \dots, N-1$, then position space photon wavepacket with a Gaussian envelop can be given by Eq (1).

$$\psi_n(x_0, m, \sigma) = A e^{-\frac{(n-x_0)^2}{2\sigma^2}} e^{2\pi i m n / N} \quad (1)$$

where, x_0 is the position of the packet and σ is the width of the packet [11]. The parameter, m is related to the momentum of the packet. Graphical representation of the wavefunction (1) is shown in Figure 1.

Hilbert Space and the Hamiltonian: As mentioned, the Hilbert space in our investigation was spanned by a set of basis functions $|u_k\rangle = \sin(kx)$, $k = 1, 2, \dots, D$. A graphical representation of the Hilbert space is given in Figure 2.

In order to represent photon wavefunction as a vector in the Hilbert space, we take Fourier transform of the wavefunction,

$$c_k = N^{-1/2} \sum_n \psi_n e^{-2\pi i k n / N}, \quad k = 1, 2, 3, \dots, N \quad (2)$$

The vector of coefficients $[c_1, c_2, c_3, \dots, c_N]^T$ represented the photon wavefunction in the Hilbert space. That is, $|\psi\rangle = [c_1, c_2, c_3, \dots, c_N]^T$. In order to reduce the computational complexity, we reduce the dimension of the Hilbert space to a smaller number D and accordingly the wavefunction became

$$|\psi\rangle = [c_1, c_2, c_3, \dots, c_D]^T$$

We identified the associated Hamiltonian operator, \hat{H} to the amplitude of the photon wavefunction. Then the matrix element of the Hamiltonian was calculated as follows,

$$\hat{H}_{kl} = \langle u_k | \hat{H} | u_l \rangle = \int_{-\infty}^{\infty} \sin(kx) \hat{H} [\sin(lx)] dx \approx \sum_{n=0}^{N-1} \sin(kn) \hat{H} \sin(ln) \quad (3)$$

Equation of Motion: The time evolution of $|\psi\rangle$ was given by the Schrodinger equation,

$$i \frac{d}{dt} |\psi\rangle = \hat{H} |\psi\rangle \quad (4)$$

In small time step Δt , the evolution of the wavefunction, $\Delta|\psi\rangle$ was approximated by,

$$\Delta|\psi\rangle = -i \hat{H} |\psi\rangle \Delta t$$

Therefore, wavefunction at time Δt was given by,

$$|\psi(t + \Delta t)\rangle = |\psi(t)\rangle + \Delta|\psi\rangle = |\psi(t)\rangle - i \hat{H} |\psi\rangle \Delta t \quad (5)$$

Simulation Results: We performed numerical simulation of time evolution of photon wavepacket by application of the Hamiltonian on the wavefunction using Eqs (4) and (5). We adopted a closed-system approach without regarding the effect of environment. As expected, the wavefunction at any fixed point on the x -axis was observed to evolve in a complex phase plane as seen in Figure 3.

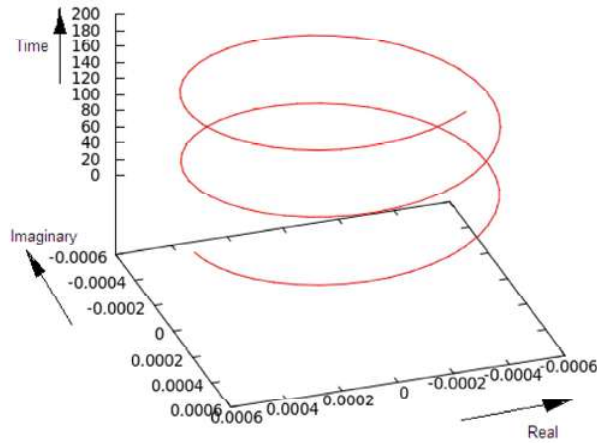


Fig. 3: Time evolution of $|\psi\rangle$ at a fixed spatial point in the complex phase-plane.

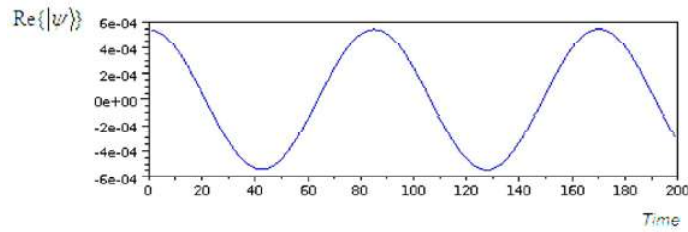


Fig. 4: Time evolution of the real part of the wavefunction $\text{Re}\{|\psi\rangle\}$ at a fixed spatial point.

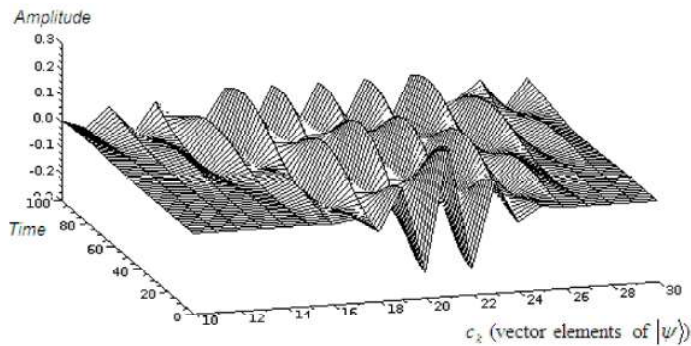


Fig. 5: Time evolution of the vector elements of $|\psi\rangle$ in the Hilbert space.

The plot of real value of the wavefunction $\text{Re}\{|\psi\rangle\}$ versus time at the same spatial point is shown in Figure 4. As seen in this figure, the real part of the wavefunction was evolving with a nearly sinusoidal pattern. Time evolution of a portion of the vector elements of $|\psi\rangle = [c_1, c_2, c_3, \dots, c_D]^T$ in the Hilbert space is shown in Figure 5. The figure shows how the vector elements were changing in time due to aforesaid evolution in the complex phase plane. Finally, time evolution of the corresponding portion of the photon packet in real-space (x-axis) is given in Figure 6. Real-space instantaneous values were found by mapping

vectors from Hilbert space to real-space as given by Eq (6).

$$\psi(x) = \sum_{k=1}^D c_k \sin(kx) \quad (6)$$

Or, equivalently as discrete points given by Eq (7)

$$\psi_n = \sum_{k=1}^D c_k \sin(kn), \quad n = 0, 1, 2, \dots, N-1 \quad (7)$$

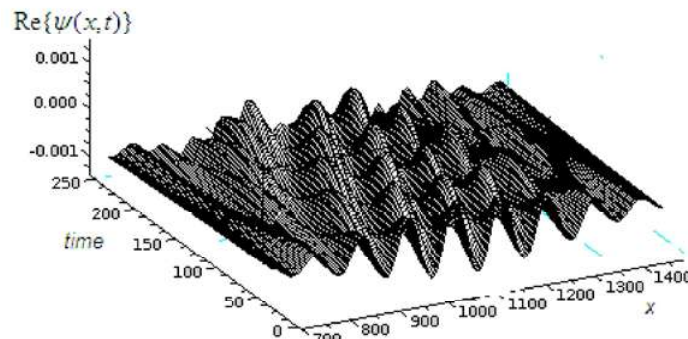


Fig. 6: Time evolution of the real-space photon wavefunction.

In Figure 6, it is clearly seen that although every point of the wavepacket *et al.* along the x-axis was evolving in the complex phase plane, the wavepacket still remained considerably localized and the overall Gaussian shape of the wavepacket was still reasonably maintained.

CONCLUSION

We studied quantum evolution of a photon wavepacket in the complex phase-plane. A suitable Hilbert space spanned by a set of sinusoidal basis functions was constructed and an appropriated Hamiltonian was devised. Real-space quantities were transformed into the Hilbert space through Fourier transform and vice-versa. Equations of evolution were derived from standard Schrodinger equation. Simulation results showed efficacy of the approach.

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