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A General Algorithm for Solving the Fractional Reaction Diffusion Model Arising in Bacterial Colony by Homotopy Analysis Transform Method

Ahmed Salah and Samia.S.A.Hassan

Mathematics and Theoretical Physics Dept.,
Nuclear Research Center, Atomic Energy Authority, P.N.13759, Egypt

Abstract: In this paper, we study the approximation of analytical solution for Fractional Reaction Diffusion model which describes the evaluation of bacterium Bacillus, which grows on the surface of thin agar plates, by using homotopy analysis transform method. The fractional derivatives are described by caputo sense. A comparative study between the homotopy analysis transform method and the classical Adomain's Decomposition Method is conducted. Results show that the homotopy analysis transform method is accurate when applied to a fractional reaction diffusion model. Numerical results with Tables and Figures are given.

Key words: Homotopy analyses transform method · Laplace transformation · Fractional Calculus · Fractional Reaction Diffusion Model · Bacteria growth

INTRODUCTION

In recent years much attention has been focused on the analytical methods, Adomian decomposition method (ADM) has been studied [1], Variational iteration method (VIM) has been applied in many problem [2], Homotopy perturbation method (HPM)[3] has been investigated, Homotopy analysis method (HAM) [4-5] properly overcomes restrictions of perturbation techniques because it does not need any small or large parameters to be contained in the problems. Salah *et al*. [6] investigated a new algorithm for solving the nonlinear problem by mean of the homotopy analysis method with addition to the Laplace transformation. The method so effective and has been applied in fuzzy heat equation [7]. This method has been used to obtain approximation solution of a large class of linear and nonlinear differential equation. It is also very essay to applied in a computer program.

In this paper we present a solution of a more general model of fractional reaction-diffusion problem has been raised in most biological system by using the new method. It so call the homotopy analysis transform method. In more general we can write the Fractional-Reaction Diffusion equation by

$$\frac{\partial^{\alpha_1} b}{\partial t^{\alpha_1}} = D_b \frac{\partial}{\partial x} nb \frac{\partial b}{\partial x} + nb$$
$$\frac{\partial^{\alpha_2} n}{\partial t^{\alpha_2}} = D_n \frac{\partial^2 n}{\partial x^2} - nb \tag{1.1}$$

As the initial conditions, we set

$b(x, 0) = b_0$

$n(x, 0) = n_0$

where D_b, D_b are the diffusion coefficients describe the bacterial movement and nutrient, $b(x, t)$ the population density $n(x, t)$ the concentration of the nutrient, n_0 is the initial concentration of the nutrient and b_0 is the density of the initial concentration.

This model has been solved by a classical Adomian Decomposition method [8], we are comparing between the classical method and the new method, also the reason of using the fractional order differential equation is that FOD are naturally related to system with memory which exists in most biological systems.

Corresponding Author: Ahmed Salah, Mathematics and Theoretical Physics Dept., Nuclear Research Center, Atomic Energy Authority, P.N.13759, Egypt.

Notations and Definitions of Fractional Calculus: A fractional derivative has received considerable interest in recent years. In many applications, fractional derivative provide more accurate models of the systems than ordinary derivatives. Many applications of fractional derivative in the areas of solid mechanics and modeling of viscoelastic damping, electrochemical processes, dielectric polarization, colored noise, bioengineering and various branches of science and engineering could be found, among others, in [9-16]. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations. Here, we mention the basic definitions of the Caputo fractional order integration and differentiation, which are used in the up coming paper and play the most important role in the theory of differential.

We begin with the Riemann-Lioville definition of the fractional integral of order $\alpha>0$, which is given as

$$I^{\alpha}y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}y(\tau)d\tau, \qquad (\alpha>0) \tag{2.1}$$

where Γ is the gamma function, for integer $\alpha>0$, Eq.1 is known as the Cushy integral formula. Here we take the lower limit of the integral as 0, however, a nonzero limit can also be taken. It can be verified that the integral operator I^{α} commutes, i.e.

$$I^{\alpha}I^{\beta}y(t)=I^{\beta}I^{\alpha}y(t)=I^{\alpha+\beta}y(t) \qquad (\alpha>0) \tag{2.2}$$

We will largely deal with Caputo fractional derivatives. However, we will also come across the Riemann Liouville fractional derivatives these two derivatives are given as:

Caputo Fractional Derivative

$$D_*^{\alpha}y(t)=I^{n-\alpha}D^{n}y(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\left(\frac{d}{d\tau}\right)^{n}y(\tau)d\tau \tag{2.3}$$

Riemann Liouville Fractional Derivative

$$D^{\alpha}y(t)=D^{n}I^{n-\alpha}y(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{d\tau}\right)^{n}\int_0^t (t-\tau)^{n-\alpha-1}y(\tau)d\tau \tag{2.4}$$

where $\alpha>0$, n is the smallest integer greater than or equal to α and the operator D^n is the ordinary differential operator. These two derivatives are related by the formula

$$D^{\alpha}y(t)=D_*^{\alpha}y(t)+\sum_{i=0}^{n-1}\frac{t^{i-\alpha}}{\Gamma(i-\alpha+1)}y^{i}(0^{+}). \tag{2.5}$$

Observe that for zero ICs, the two derivatives are the same. Thus, for this condition we may switch between the two derivatives as necessary.

Homotopy Analysis Transform Method: We consider the approximation solution of nonlinear fractional equation by using HATM [7], we can write the system in the form

$$\frac{\partial^{\alpha_i}}{\partial t^{\alpha_i}}u_i(x,t)=N_i\left[u_i(x,t)\right]+f_i(x,t) \tag{3.1}$$

With the initial condition

$$u_i\,(x,\,0)=u_{0i}(x) \tag{3.2}$$

where $\partial^{\alpha_i}/\partial t^{\alpha_i}$ are the caputo derivate of order $\alpha_i(i$ = 1,2,...........,$k)$, N_i are nonlinear operator. $u_i(x,\,t)$ is unknown functions and $(x,\,t)$ are independent variable can be written as

$$\frac{\partial}{\partial t}u_i(x,t)=\frac{\partial^{1-\alpha_i}}{\partial t^{1-\alpha_i}}\left[N_i\left[u_i(x,t)\right]\right]+\frac{\partial^{1-\alpha_i}}{\partial t^{1-\alpha_i}}\left[f_i(x,t)\right] \tag{3.3}$$

Applying Laplace transformation

$$\Im\left[u_i(x,t)\right]-\frac{u_i(x,0)}{s}=\frac{1}{s}\Im\left[\frac{\partial^{1-\alpha_i}}{\partial t^{1-\alpha_i}}\left[N_i\left[u_i(x,t)\right]\right]\right]+\frac{1}{s}\Im\left[\frac{\partial^{1-\alpha_i}}{\partial t^{1-\alpha_i}}\left[f_i(x,t)\right]\right] \tag{3.4}$$

Equation (3.4) can be written as a nonlinear operator from as follow

$$\tilde{N}\left[u_i(x,t)\right]=0 \tag{3.5}$$

where $\Im$ an Laplace transformation, we have a nonlinear operator

$$\tilde{N}\left[u_i(x,t)\right]=\Im\left[u_i(x,t)\right]-\frac{u_i(x,0)}{s}-\frac{1}{s}\Im\left[\frac{\partial^{1-\alpha_i}}{\partial t^{1-\alpha_i}}\left[N_i\left[u_i(x,t)\right]\right]\right]-\frac{1}{s}\Im\left[\frac{\partial^{1-\alpha_i}}{\partial t^{1-\alpha_i}}\left[f_i(x,t)\right]\right] \tag{3.6}$$

where q ϵ[0, 1] an embedding parameter and $\hbar \neq 0$ an auxiliary parameter, we can construct the zeroth-order deformation equation [4]

$$(1-q)\Im\left[\tilde{\phi}_i(x,t;q)-u_{0i}(x,t)\right]=q\hbar\tilde{N}_i\left[\tilde{\phi}_i(x,t;q)\right] \tag{3.7}$$

When q = 0 and q = 1, the zero-order deformation equation become

$$\tilde{\phi}_i(x,t;0)=u_{0i}(x,t),\ \tilde{\phi}_i(x,t;1)=u_i(x,t) \tag{3.8}$$

By Taylor's theorem, $\tilde{\phi}_i(x,t;q)$ can be expand in a power series of q as follows

$$\tilde{\phi}_i(x,t;q)=u_{0i}(x,t)+\sum_{m=1}^{\infty}u_{im}(x,t)q^m \tag{3.9}$$

where

$$u_{im}(x,t)=\frac{1}{m!}\frac{\partial^m\tilde{\phi}_i(x,t;q)}{\partial q^m}\Bigg|_{q=0} \tag{3.10}$$

By using the zeroth order equation (3.7) and setting q= 1, we have the mth order deformation

$$\Im\left[u_{im}(x,t)-\chi_m u_{im-1}(x,t)\right]=\hbar R_{im}\left[u_{im-1}(x,t)\right] \tag{3.11}$$

By applying inverse Laplace transform Eq(3.11) becomes

$$u_{im}(x,t)=\chi_m u_{im-1}(x,t)+\hbar\Im^{-1}R_{im}\left[u_{im-1}(x,t)\right] \tag{3.12}$$

where

$$R_{im}\left[u_{im-1}(x,t)\right]=\frac{1}{(m-1)!}\frac{\partial^{m-1}\tilde{N}_i\left[\tilde{\phi}_i(x,t;q)\right]}{\partial q^{m-1}}\Bigg|_{q=0} \tag{3.13}$$

$$\chi_m=\begin{cases}0 & m<0\\ 1 & m\geq 0\end{cases} \tag{3.14}$$

Application and Numerical Results: We consider the nonlinear fractional reaction diffusion model for arising in the bacterial growth written as

$$\begin{aligned}\frac{\partial^{\alpha_1}b}{\partial t^{\alpha_1}}&=D_b\frac{\partial}{\partial x}nb\frac{\partial b}{\partial x}+nb\\ \frac{\partial^{\alpha_2}n}{\partial t^{\alpha_2}}&=D_n\frac{\partial^2 n}{\partial x^2}-nb\end{aligned} \tag{4.1}$$

Appling the inverses operators $\frac{\partial^{1-\alpha_1}}{\partial t^{1-\alpha_1}}$ and $\frac{\partial^{1-\alpha_2}}{\partial t^{1-\alpha_2}}$ to the system

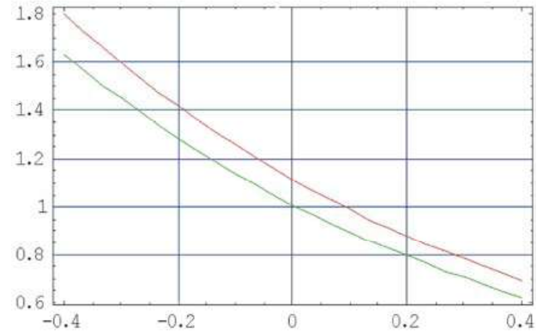
$$\begin{aligned}\frac{\partial b}{\partial t}&=D_b\frac{\partial^{1-\alpha_1}}{\partial t^{1-\alpha_1}}\left\{\frac{\partial}{\partial x}nb\frac{\partial b}{\partial x}\right\}+\frac{\partial^{1-\alpha_1}}{\partial t^{1-\alpha_1}}\{nb\}\\ \frac{\partial n}{\partial t}&=D_n\frac{\partial^{1-\alpha_2}}{\partial t^{1-\alpha_2}}\left\{\frac{\partial^2 n}{\partial x^2}\right\}-\frac{\partial^{1-\alpha_2}}{\partial t^{1-\alpha_2}}\{nb\}\end{aligned} \tag{4.2}$$

Take Laplace transform on the system and subject the initial conditions

$$\begin{aligned}s\Im\{b\}-b_0&=D_b\Im\left\{\frac{\partial^{1-\alpha_1}}{\partial t^{1-\alpha_1}}\left\{\frac{\partial}{\partial x}nb\frac{\partial b}{\partial x}\right\}\right\}+\Im\left\{\frac{\partial^{1-\alpha_1}}{\partial t^{1-\alpha_1}}\{nb\}\right\}\\ s\Im\{n\}-sn_0&=D_n\Im\left\{\frac{\partial^{1-\alpha_2}}{\partial t^{1-\alpha_2}}\left\{\frac{\partial^2 n}{\partial x^2}\right\}\right\}-\Im\left\{\frac{\partial^{1-\alpha_2}}{\partial t^{1-\alpha_2}}\{nb\}\right\}\end{aligned} \tag{4.3}$$

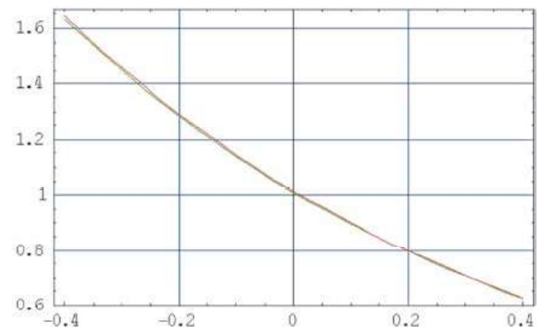
Eq (4.3) can be written as a nonlinear operator from as follow:

$$\begin{aligned}\tilde{N}_1[\phi_1]&=\Im\{\phi_1\}-\frac{b_0}{s}-\frac{1}{s}D_b\Im\left\{\frac{\partial^{1-\alpha_1}}{\partial t^{1-\alpha_1}}\left\{\frac{\partial}{\partial x}\phi_2\phi_1\frac{\partial\phi_1}{\partial x}\right\}\right\}-\frac{1}{s}\Im\left\{\frac{\partial^{1-\alpha_1}}{\partial t^{1-\alpha_1}}\{\phi_2\phi_1\}\right\}\\ \tilde{N}_2[\phi_2]&=\Im\{\phi_2\}-\frac{n_0}{s}-\frac{1}{s}D_n\Im\left\{\frac{\partial^{1-\alpha_1}}{\partial t^{1-\alpha_1}}\left\{\frac{\partial^2\phi_2}{\partial x^2}\right\}\right\}+\frac{1}{s}\Im\left\{\frac{\partial^{1-\alpha_1}}{\partial t^{1-\alpha_1}}\{\phi_2\phi_1\}\right\}\end{aligned} \tag{4.4}$$



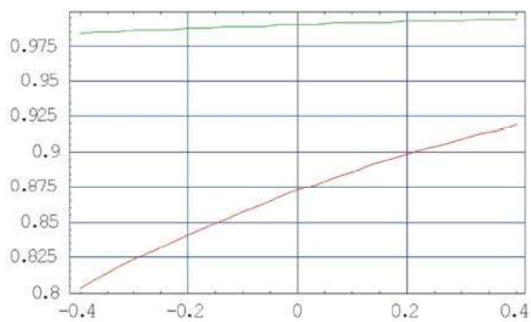
Bacteria growth at $b(x, 0.01)$

Fig. 1: The numerical results at $a = 1.2$, $D_b = 0.01$, $t = 0.01$, $\alpha_1 = \alpha_2 = 1$ (red line), $\alpha_1 = \alpha_2 = 0.5$ (green line)



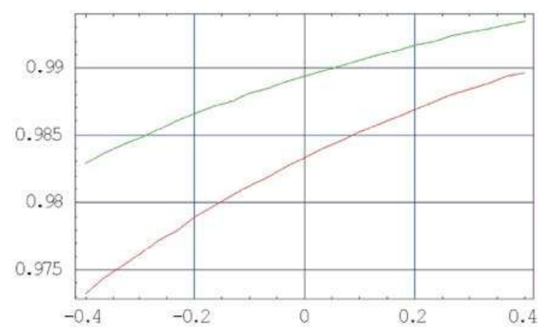
Bacteria growth at $b(x, 0.01)$

Fig. 2: The numerical results at $a = 1.2$, $D_b = 0.01$, $t = 0.01$, $\alpha_1 = \alpha_2 = 99$ (red line), $\alpha_1 = 1$, $\alpha_2 = 0.9$ (green line)



Concentration of nutrient $n(x, 0.01)$

Fig. 3: The numerical results at $a = 1.2$, $D_b = 0.01$, $t = 0.01$, $D_n = 1$ $\alpha_1 = \alpha_2 = 1$ (red line), $\alpha_1 = \alpha_2 = 0.5$ (green line)



Concentration of nutrient $n(x, 0.01)$

Fig. 4: The numerical results at $a = 1.2$, $D_b = 0.01$, $t = 0.01$, $D_n = 1$ $\alpha_1 = \alpha_2 = .99$ (red line), $\alpha_1 = 1$, $\alpha_2 = 0.9$ (green line)

By mean of Zeroth deformation equation

$$
\begin{aligned}
b_m(x,t) &= \chi_m b_{m-1}(x,t) - \Im^{-1}\Re_1\left(b_{m-1}(x,t)\right) \\
n_m(x,t) &= \chi_m n_{m-1}(x,t) - \Im^{-1}\Re_2\left(n_{m-1}(x,t)\right)
\end{aligned}
\tag{4.5}
$$

where

$$
\begin{aligned}
\Re_1\left(b_{m-1}\right) &= \Im\{b_{m-1}\} - \frac{b_0}{s} - \frac{1}{s} D_b \Im\left\{\frac{\partial^{1-\alpha_1}}{\partial t^{1-\alpha_1}}\left\{\frac{\partial}{\partial x} A_{m-1}\right\}\right\} - \frac{1}{s}\Im\left\{\frac{\partial^{1-\alpha_1}}{\partial t^{1-\alpha_1}}\{B_{m-1}\}\right\} \\
\Re_2\left(n_{m-1}\right) &= \Im\{n_{m-1}\} - \frac{n_0}{s} - \frac{1}{s} D_n \Im\left\{\frac{\partial^{1-\alpha_1}}{\partial t^{1-\alpha_1}}\left\{\frac{\partial^2 n_{m-1}}{\partial x^2}\right\}\right\} + \frac{1}{s}\Im\left\{\frac{\partial^{1-\alpha_1}}{\partial t^{1-\alpha_1}}\{B_{m-1}\}\right\}
\end{aligned}
\tag{4.6}
$$

With

$$
\begin{aligned}
A_{m-1} &= \sum_{j}^{m-1} \frac{\partial b_j}{\partial x} B_{m-j-1} \\
B_{m-1} &= \sum_{i}^{m-1} n_i b_{m-i-1}
\end{aligned}
\tag{4.7}
$$

Subject the initial conditions

$$b_0(x,t) = e^{-ax}$$
$$n_0(x,t) = 1 \tag{4.8}$$

And by using (4.5)-(4.8) we shall be able to calculate some of terms of the convergent series as

$$b_0(x,t) = e^{-ax}$$
$$b_1(x,t) = (2a^2 e^{-ax} + 1)e^{-ax}\frac{t^{\alpha_1}}{\Gamma(\alpha_1 + 1)}$$
$$b_2(x,t) = (18a^4 D_b^2 e^{-2ax} + 6a^2 D_b e^{-ax} + 1)e^{-ax}\frac{t^{\alpha_2}}{\Gamma(\alpha_2 + 1)} - (3a^2 e^{-ax} + 1)e^{-ax}\frac{t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1 + \alpha_2 + 1)} \tag{4.9}$$

And

$$n_0(x,t) = 1$$
$$n_1(x,t) = e^{-ax}\frac{t^{\alpha_2}}{\Gamma(\alpha_2 + 1)}$$
$$n_2(x,t) = (e^{-ax} - a^2 D_b)e^{-ax}\frac{t^{\alpha_2}}{\Gamma(\alpha_2 + 1)} - (2a^2 e^{-ax} + 1)e^{-ax}\frac{t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1 + \alpha_2 + 1)} \tag{4.10}$$

Table 1: Bacteria growth $b(x, t)$ in case a = 1.2, D_b = 0.01, t = 0.01

x	$\alpha_1 = 1$ $\alpha_1 = 1$ ADM[9]	$\alpha_1 = 1$ $\alpha_1 = 1$ HATM	$\alpha_1 = .5$ $\alpha_1 = .5$ ADM	$\alpha_1 = .5$ $\alpha_1 = .5$ HATM
-0.4	1.632950	1.63295541	0.773	0.773
-0.2	1.2844171	1.28441117	1.28441	1.28441
0	1.010293	1.0102921	1.01029	1.01029
0.2	0.7946835	0.79468211	0.794682	0.794682
0.4	0.6250950	0.625094951	0.625094	0.625094

Table 1: Concentration of nutrient $n(x, t)$ in case a = 1.2, D_b = 0.01, t = 0.01

x	$\alpha_1 = 1$ $\alpha_1 = 1$ ADM[9]	$\alpha_1 = 1$ $\alpha_1 = 1$ HATM	$\alpha_1 = 1$ $\alpha_1 = 1$ ADM	$\alpha_1 = 1$ $\alpha_1 = 1$ HATM
-0.4	0.773	0.773	0.773	0.773
-0.2	1.28441	1.28441	1.28441	1.28441
0	1.01029	1.01029	1.01029	1.01029
0.2	0.794682	0.794682	0.794682	0.794682
0.4	0.625094	0.625094	0.625094	0.625094

CONCLUSION

In this paper, the homotopy analysis transform method was applied to solve fractional reaction diffusion model which describe the evolution of the bacterium growth.

The result show that the solution continuously depends on time-fractional derivative. we give the solution in a convergence series. It provides series solutions which generally converge very rapidly in real physics phenomena.

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