World Applied Sciences Journal 19 (11): 1649-1652, 2012
ISSN 1818-4952

DOI: 10.5829/idosi.wasj.2012.19.11.1872

Permuting Tri-Derivations on Incline Algebras

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Abstract: In this paper, we introduce the notion of a permuting tri-derivation on an incline algebra and investigate some related properties. Also using this notion we characterize distributive element of an incline.

Mathematics Subject Classification (2010): 16Y60, 16W25.
Key words: Incline · Integral incline · Derivation · Distributive element · Permutin tri-derivation

INTRODUCTION

The concept of an incline was introduced by Cao [1] and later developed by Cao, *et al.* [1]. Recently a survey on inclines has been made by Kim and Roush [2]. An Incline algebra is a generalization of both Boolean and fuzzy algebras and also is a special type of a semiring. It has a semiring structure as well as a poset structure. It can also be used to represent automata and other mathematical systems, in optimization theory and the study of inequalities for non-negative matrices of polynomials. It is known that the two sided ideals of a ring form an incline.

Derivations on various algebraic structures have been an active area of research since the last fifty years due to their usefulness in various areas of mathematics. The notion of a derivation on a ring plays an important role in the characterization of rings. The notion of derivation on a lattice was defined and some of its related properties were examined firstly by Szasz [3]. Ozturk [4] introduced the notion of a permuting tri-derivation in prime and semiprime rings and proved some results. Later on Ozturk, Yazarli and Kim [5] applied this notion to lattices. As generalizations of derivations, f- derivations, (f, g)-derivations, symmetric bi-derivations, permuting tri-derivations, symmetric f -biderivations, permuting f-triderivations, on prime, semiprime rings and lattices are studied by a lot of researchers [6, 7, 4, 5, 3] and therein.

The notion of a derivation for an incline algebra was introduced by Al-Shehri in [8]. In this paper he also discussed some of its properties. In this paper, as a generalization of a derivation of an incline algebra, the notion of a permuting tri-derivation on an incline algebra is introduced. Further using this notion some properties of an incline algebra are also investigated.

Preliminaries: Inclines are additively idempotent semirings in which products are less than (or) equal to either factor. An incline algebra is a non- empty set R with binary operations, denoted by + and *, satisfying the following axioms for all $x, y, z \in R$:

(I) $x + y = y + x$,
(I I) $x + (y + z) = (x + y) + z$,
(I I I) $x * (y * z) = (x * y) * z$,
(I V) $x * (y + z) = (x * y) + (x * z)$,
(V) $(y + z) * x = (y * x) + (z * x)$,
(V I) $x + x = x$,
(V I I) $x + (x * y) = x$,
(V I I I) $y + (x * y) = y$.

Furthermore, an incline algebra R is said to be commutative if $x * y = y * x$ for all $x, y \in R$. For convenience, we will say + (*) as addition (multiplication).

Moreover, Incline theory is based on semiring theory and lattice theory. Every distributive lattice is an incline. An incline is a distributive lattice (as a semiring) if and only if $x * x = x$ for all $x \in R$ ([9], Proposition 1.1.1). A subincline of an incline R is a nonempty subset M of R which is closed under addition and multiplication.

In an incline we define $x \leq y$ if and only if $x + y = y$ for all $x, y \in R$. It is easy to see that $\leq$ is a partial order on R and that for $x, y \in R$, *the* element $x + y$ is the least upper bound of $\{x, y\}$. We say that $\leq$ is induced by operation +. It follows that

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- $x * y \le x$ and $x * y \le y$ for all $x, y \in R$,
- $y \le z$ implies $x * y \le x$ and $y * x \le z * x$ for any $x, y, z \in R$,
- If $x \le y$, $a \le b$, then $x + a \le y + b$, $x * a \le y * b$.

An ideal in an incline R is a subincline $M \in R$ such that if $x \in R$ and $y \le x$ then $y \in M$. An element 0 in an incline algebra R is a zero element if $x + 0 = x = 0 + x$ and $x * 0 = 0 * x = 0$ for $x \in R$. An element $1 \neq$ zero element in an incline algebra R is called a multiplicative identity if for $x \in R$, $x * 1 = 1 * x = x$. A non-zero element a in an incline algebra R with a zero element is said to be a left (resp. right) zero divisor if there exists a non-zero element $b \in R$ such that $a * b = 0$ (resp. $b * a = 0$). A zero divisor is an element of R which is both a left zero divisor and a right zero. An incline algebra R with a multiplicative identity 1 and a zero element 0 is called an integral incline if it has no zero divisors.

Let $(R, +,*)$ be an incline algebra and $D: R \to R$ a mapping. Then D is called a derivation on R if $D(x * y) = (D(x) * y) + (x * D(y))$ for all $x, y \in R$. Let $(R, +,*)$ be an incline algebra. A mapping $D(., ., .): R \times R \times R \to R$ is called a permuting mapping if $D(x, y, z) = D(x, z, y) = D(y, x, z) = D(y, z, x) = D(z, x, y) = D(z, y, x)$ for all $x, y, z \in R$. Let $(R, +, *)$ be an incline algebra. A mapping $d: R \to R$ defined by $d(x) = D(x, x, x)$ is called the trace of $D(.,.,.)$, where $D(.,.,.): R \times R \times R \to R$ is a permuting mapping.

RESULTS

Definition 1: *Let $(R, +,*)$ be an incline algebra and $D: R \times R \times R \to R$ a permuting mapping, then D is said to be a permuting tri-derivation on R if it satisfies the following condition:*

$D(x * w, y, z) = (D(x, y, z) * w) + (x * D(w, y, z))$ *for all* $w, x, y, z \in R$.

Remark 1: It is obvious that if D is a permuting tri-derivation on R then it satisfies the relations $D(x, y * w, z) = (D(x, y, z) * w) + (y * D(x, w, z))$ and $D(x, y, z * w) = (D(x, y, z) * w) + (z * D(x, y, w))$ for all $w, x, y, z \in R$.

Example 1: Let $(R, +,*)$ be a commutative incline algebra and $D: R \times R \times R \to R$ a permuting mapping defined by $D(x, y, z) = (x * y) * z$ for all $x, y, z \in R$. Then it is easy to verify that D is a permuting tri-derivation on R.

Example 2: Let $(R, +,*)$ be a commutative incline algebra and $D: R \times R \times R \to R$ a permuting mapping such that for an element $a \in R$, $D(x, y, z) = a * ((x * y) * z)$ for all $x, y, z \in R$. Then it is easy to verify that D is a permuting tri-derivation on R.

Proposition 1: Let $(R, +,*)$ *be an incline algebra and D a permuting tri-derivation on R. Then the following hold for all $w, x, y, z \in R$:*

(*i*) $D(x * w, y, z) \le D(x, y, z) + D(w, y, z)$,
(*ii*) *If* $x \le w$, *then* $D(x * w, y, z) \le w$.

Proof: (*i*) Let $w, x, y, z \in R$. Using (1) we have $D(x, y, z) * w \le D(x, y, z)$ and $x * D(w, y, z) \le D(w, y, z)$.

Using (3) we get, $(D(x, y, z) * w) + (x * D(w, y, z)) \le D(x, y, z) + D(w, y, z)$.
That is, $D(x * w, y, z) = D(x, y, z) + D(w, y, z)$.

(*ii*) Using (1) we get the relation $D(x, y, z) * w \le w$. Let $x \le w$.

Using (1) and (3), we get $x * D(w, y, z) \le w * D(w, y, z) \le w$.

That is, $D(x * w, y, z) = (D(x, y, z) * w) + (x * D(w, y, z)) \le w + w$. Hence $D(x * w, y, z) \le w$.

Proposition 2: *Let $(R, +,*)$ be a commutative incline algebra and D a permuting tri-derivation on R. If R is a distributive lattice, then $D(x, y, z) \le x$, $D(x, y, z) \le y$ and $D(x, y, z) \le z$.*

Proof: Let R be a distributive lattice. Then $D(x, y, z) = D(x * x, y, z) = (D(x, y, z) * x) + (x * D(x, y, z))$, which gives $D(x, y, z) + x = (D(x, y, z) * x) + (x * D(x, y, z)) + x = (D(x, y, z) * x) + (D(x, y, z) * x) + x$. That is, $D(x, y, z) + x = (D(x, y, z) * x) + x$. Using ($VIII$), we get $D(x, y, z) + x = x$. Thus $D(x, y, z) \le x$. Similarly one can get $D(x, y, z) \le y$ and $D(x, y, z) \le z$.

Corollary 1: Let $(R, +,*)$ *be an incline algebra with a multiplicative identity and d the trace of the permuting tri-derivation D of R.*

Then x * D(1, $y, z) \le D(x, y, z)$.

Proof: Let $x \in R$. Then $D(x, y, z) = D(x * 1, y, z) = (D(x, y, z) * 1) + (x * D(1, y, z)) = D(x, y, z) + (x * D(1, y, z))$. That is, $x * D(1, y, z) \le D(x, y, z)$.

Proposition 3: Let $(R, +, *)$ *be an integral incline and* D *a permuting tri-derivation of* R. *If* a *is an element of* R *and* $a * D(x, y, z) = 0$, *then* $a = 0$ *or* $D = 0$.

Proof: Let $a * D(x, y, z) = 0$ for all $x, y, z \in R$. Let $w \in R$. Replacing x by $x * w$ in the last relation, we get $0 = a * D(x * w, y, z)$. That is, $0 = (a * (D(x, y, z) * w)) + (a * (x * D(w, y, z)))$. Using hypothesis from the last relation we get $0 = a * (x * D(w, y, z))$. Replacing $x = 1$ in the last relation we get $a * D(w, y, z) = 0$. Since R is an integral incline, therefore either $a = 0$ or $D = 0$.

Definition 2: *Let* $(R, +, *)$ *be an incline algebra and* $D: R \times R \times R \to R$ *a permuting mapping. We call* D *a joinitive permuting tri- derivation if*

$$D(x + w, y, z) = D(x, y, z) + D(w, y, z) \text{ for all } w, x, y, z \in R.$$

Theorem 1: *Let* $(R, +, *)$ *be an integral incline algebra. If* D_1 *and* D_2 *are joinitive permuting tri-derivations on* R *such that* $D_1(d_2(x), x, x) = 0$, *where* d_1 *and* d_2 *are the traces of* D_1 *and* D_2 *respectively, then either* $d_1 = 0$ *or* $d_2 = 0$.

Proof: Let $D_1(d_2(x), x, x) = 0$ for all $x \in R$. Using (VII), we get $0 = D_1(d_2(x) + (d_2(x) * x), x, x)$. That is, $0 = D_1(d_2(x), x, x) + (D_1(d_2(x), x, x) * x) + d_2(x) * D_1(x, x, x)$. Using (VII) from the last relation we get $0 = D_1(d_2(x), x, x) + d_2(x) * d_1(x)$. Using hypothesis, we get $0 = d_2(x) * d_1(x)$ for all $x \in R$. Since R is an integral incline, therefore either $d_1 = 0$ or $d_2 = 0$.

Corollary 2: *Let* $(R, +, *)$ *be an integral incline algebra and* D *a joinitive permuting tri-derivation on* R *with trace* d. *If* $D(d(x), x, x) = 0$, *then* $d = 0$.

Proof: The proof is obvious.

Theorem 2: *Let* $(R, +, *)$ *be an incline algebra and* d *the trace of the joinitive permuting tri-derivation* D *of* R. *Then*

$$d(x + y) = d(x) + d(y) + D(x, x, y) + D(x, y, y) \text{ and}$$
$$d(x) + d(y) \le d(x + y).$$

Proof: Let $x, y \in R$. Then $d(x + y) = D(x + y, x + y, x + y)$. That is, $d(x + y) = D(x, x + y, x + y) + D(y, x + y, x + y)$, which implies $d(x+y) = D(x, x, x+y)+D(x, y, x+y)+D(y, x, x+y)+D(y, y, x+y)$. The last relation gives $d(x+y) = D(x, x, x)+D(x, x, y)+D(x, y, x)+D(x, y, y) + D(y, x, x) + D(y, x, y) + D(y, y, x) + D(y, y, y)$. Thus $d(x + y) = d(x) + d(y) + D(x, x, y) + D(x, y, y)$, which gives $d(x) + d(y) \le d(x + y)$.

Corollary 3: *Let* $(R, +, *)$ *be an incline algebra and* d *the trace of joinitive permuting tri-derivation* D *of* R. *Then* $D(x * y, x, x) \le d(x)$, *for all* $x, y \in R$.

Proof: Let $x, y \in R$. Using (VII), we get $d(x) = D(x, x, x) = D(x + (x * y), x, x) = D(x, x, x) + D(x * y, x, x)$.
Thus $D(x * y, x, x) = d(x)$.

Proposition 4: *Let the commutative incline algebra* $(R, +, *)$ *be a distributive lattice and* D *a permuting tri-derivation on* R *with trace* d, *then* $d(x) \le x$, *for all* $x \in R$.

Proof: Let $x \in R$. Then $d(x) = D(x, x, x) = D(x * x, x, x)$. That is, $d(x) = (D(x, x, x) * x) + (x * D(x, x, x))$, which implies $d(x) = (x * D(x, x, x))$. This gives $d(x) = x * d(x)$. Hence $d(x) \le x$.

Theorem 3: *Let the commutative incline algebra* $(R, +, *)$ *be a distributive lattice and* D *a permuting tri-derivation on* R *with trace* d. *Then* $d(x * y) = (d(x) * y) + (d(y) * x) + (x * y) * \{D(x, x, y) + D(x, y, y)\}$ *for all* $x, y \in R$.

Proof: Let $x, y \in R$. Then $d(x * y) = D(x * y, x * y, x * y)$. That is, $d(x * y) = (D(x, x * y, x * y) * y) + (x * D(y, x * y, x * y))$, which implies $d(x * y) = [\{(D(x, x, x * y) * y) + (x * D(x, y, x * y))\} * y] + [x * \{(D(y, x, x * y) * y) + (x * D(y, y, x * y))\}]$. From the last relation we get $d(x * y) = \{D(x, x, x * y) * (y * y)\} + \{D(x, y, x * y) * (x * y)\} + \{D(y, x, x * y) * (x * y)\} + \{(x * x) * D(y, y, x * y)\}$, which implies $d(x * y) = \{D(x, x, x * y) * y\} + \{D(x, y, x * y) * (x * y)\} + \{D(y, x, x * y) * (x * y)\} + \{x * D(y, y, x * y)\}$. The last relation gives $d(x * y) = [\{(D(x, x, x) * y) + (x * D(x, x, y))\} * y] + [\{(D(x, y, x) * y) + (x * D(x, y, y))\} * (x * y)] + [\{(D(y, x, x) * y) + (x * D(y, x, y))\} * (x * y)] + [x * \{(D(y, y, x) * y) + (x * D(y, y, y))\}$. That is, $d(x * y) = \{d(x) * (y * y)\} + \{D(x, x, y) * (x * y)\} + \{D(x, y, x) * (y * (x * y))\} + \{D(x, y, y) * (x * (x * y))\} + \{D(y, x, x) * (y * (x * y))\} + \{D(y, x, y) * (x * (x * y))\} + \{D(y, y, x) * (x * y)\} + \{(x * x) * d(y)\}$, which implies $d(x * y) = (d(x) * y) + \{D(x, x, y) * (x * y)\} + \{D(x, y, x) * (x * y)\} + \{D(x, y, y) * (x * y)\} + \{D(y, x, x) * (x * y)\} + \{D(y, x, y) * (x * y)\} + \{D(y, y, x) * (x * y)\} + (x * d(y))$. Thus $d(x * y) = (d(x) * y) + (d(y) * x) + (x * y) * \{D(x, x, y) + D(x, y, y)\}$.

Definition 3: *Let* $(R, +, *)$ *be an incline algebra. An element* a *of* R *is said to be distributive if* $a * (x + y) = (a * x) + (a * y)$, *for all* $x, y \in R$.

Theorem 4: *Let (R, +,*) be a commutative incline algebra and r an element of R. Let D: $R \times R \times R \to R$ be a permuting tri-derivation on R defined by $D(x, y, z) = r * (x * (y * z))$ for all $x, y, z \in R$. Then r is distributive if and only if D is joinitive.*

Proof: Let D be joinitive. By definition of D, we have $D(x + w, y, z) = r * ((x + w) * (y * z))$. Since D is joinitive, therefore $D(x, y, z) + D(w, y, z) = r * (x * (y * z)) + r * (w * (y * z))$. Hence $r * ((x + w) * (y * z)) = r * (x * (y * z)) + r * (w * (y * z))$. Thus r is distributive. Conversely let r be distributive. Then $r * ((x + w) * (y * z)) = r * (x * (y * z)) + r * (w * (y * z))$, which alongwith definition of D implies. $D(x + w, y, z) = D(x, y, z) + D(w, y, z)$. Hence D is joinitive.

ACKNOWLEDGMENT

The authors are thankful to the Bahauddin Zakariya University, Multan and Higher Education Commission of Pakistan for the support and facilities provided during this research.

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