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# Economic-Mathematical Model of Fair Distribution of Resource with Non-Symmetric Parameters

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**Abstract:** The problem of fair distribution of a resource which is one of the main fundamental problems of economy is considered. Game-theoretic methods investigate the model of negotiations connected with a problem of the best choice. The three members of negotiations divide cake. At each stage, members receive random offers that accept or reject. We consider economic model of negotiations with the arbitrator who applies casual offers with Dirichlet distribution. The final decision is defined by consensus. The process continues until the players reach an agreement, or until the negotiations have reached the last stage. The optimal behavior of the players is derived in the class of threshold strategies.

**Key words:** Cake division model · Consensus · Discounting · Dirichlet distribution · Multistage procedure · random offers · Threshold strategy

## INTRODUCTION

The problem of fair distribution of a resource is one of the main fundamental problems of economy. Problems of optimization of distribution of resources should be considered by means of game-theoretic methods as participants of negotiations face the conflict of interests.

The cake division problem is a classic problem in the theory of negotiations. The word "cake" is meant as any resource which has to be divided into any parts taking into account interests of the parties: distribution of territories, material values, spheres of influence, etc. Fair division or the cake-cutting problem, is the problem of dividing a resource in such a way that all negotiators believe that they have received a fair amount. The problem is easier when recipients have different measures of value of the parts of the cake. The cake-cutting problem had been one of the most important open problems in 20th century. There are various procedures of division of the cake [1-3]. The presence of different measures opens a vast potential for many challenging questions and directions of further research.

For example, the conflict of a type of fir division arises at the property section. Participants of the conflict consider the decision fair only when they consider that received the better lot or at least one of equal shares. To convince one of the parties of honesty of a sharing rather difficult, especially of a conflict situation. It is known that any attempts to divide property can easily lead to emergence of feeling of injustice.

All existing models of division of the cake can be divided into two groups. In the first group participants offer options of division of cake [4-6]. In other group for the solution of a task the third independent party – the arbitrator who forms offers to participants is invited [7-10]. Note also that arbitration procedures have anticorruption character. The arbitrator can be modeled by the generator of random numbers and design is easy to realize in the form of numerical procedure on the computer or in an information network.

All methods of cake division are fair if all participants estimate different parts of cake equally. There are basic principles of justice at the cake section for two players.

**Lack of Discrimination:** Each of $n$ of participants is sure that received not less 1/n parts of cake.

**Lack of Envy:** Each of participants is sure that received not less other participants of negotiations and therefore doesn't envy another.

**Optimality Across Pareto:** At such procedure there is no opportunity to increase a piece of one of participants that thus remained participants of negotiations remained are happy with the pieces.

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However already for three participants the principle of lack of envy for this procedure is already broken.

This paper continues the model intended to the cake division models with arbitration. We consider the situation that three players have to divide a cake of size 1.

**Problem (The Main Part):** Multistep procedure of uniform cake division of the single size for *n* persons is considered. For fair cake division is invited the arbitrator who is presented by a random number generator.

Let we have *K* steps for negotiations and up to the end there was *k* of steps. In each step the arbitrator generates random offers and presents them to participants of negotiations. The players observe their offers and decide to accept the offer or to reject it. Then it is considered number of the participants who have agreed with the proposal of the arbitrator. If it is more or equally than some set number of *p*, the decision is made. Otherwise, this version of proposals of the arbitrator is rejected and players pass to the following step where the new option is offered to them. After each step size of the cake is discounted by the value $\delta$, where $\delta<1$. If as a result of negotiations there came $k=0$ step and participants didn't come to any decision, they receive pieces of cake of the small size - some size *b*.

In article [7] the arbitration task for three players and its generalization for *n* of players where the decision is defined by vote is described. In work [11] the symmetric case of the section of cake where at each stage participants of negotiations receive offers $x^k$, $y^k$, $z^k$ for parameters $k_i=1$ $(i=1,2,3)$ is investigated. Assume that in each step the offers are random variables with the Dirichlet distribution, i.e. the density is

$$f(x,y,z)=\frac{\Gamma(k_1+k_2+k_3)}{\Gamma(k_1)\cdot\Gamma(k_2)\cdot\Gamma(k_3)}\cdot x^{k_1-1}\cdot y^{k_2-1}\cdot z^{k_3-1} \quad (1)$$

such that $x+y+z=1$. For the decision the rule of the majority or by consensus is used.

If participants of a sharing have equal weight, parameters of Dirichlet distribution should be chosen equal. Then procedure of a sharing guarantees equal opportunities for all participants. If any of participants has bigger weight, it is necessary to increase its parameter in Dirichlet distribution.

Development of these models is presented in work [12] in which the similar scheme with parameters $k_1=k_2=k_3=2$ is investigated.

To this work the work concerning a problem of the best choice [8] where the multistep scheme of casual offers also was used also is close.

Hamers [13] showed that 2-person cake sharing games always admit a unique Nash equilibrium. In note [14] proved that a cake can be divided fairly among *n* people, although each may have a different opinion as to which parts of the cake are most valuable.

**Non-Symmetric Case:** Multistep procedure of division of cake of the single size for three persons is considered. Participants of negotiations – players of *I*, *II* and *III* – on each step receive from the arbitrator of the offer $x^{k_1}, y^{k_2}, z^{k_3}$, respectively. These are the random variables distributed on the law to Dirikhlet (1) at $k_1=m$, $k_2=k_3=1$. Then

$$f(x,y)=m(m+1)x^{m-1}, \quad x+y\le 1, \quad x,y\ge 0$$

Everyone players decide to accept the offer or to reject it. The final decision is made by consensus. If all players accept their offers then, there is a shearing $(x,y,z)$. Otherwise, the proposal is ignored and the players come in to the next step *k-1* hoping for the best offer in the future. Thus there is a discounting and on the following step players divide size cake $\delta<1$. Bargaining continues until some player accepts the offer or the time of bargaining ends.

Denote $H_k$ be the value of the game where k steps are left to go. Suppose that each player is informed only about the value of her offer. Let *x,y,z* are the offers of players *I*, *II*, *III*, respectively and as $x+y+z=1$, that can be limited model consideration by variables *x* and *y* (where $z=1-x-y$).

Let's enter into strategy consideration. Let $\mu_1(x)$ is a probability of that the player of *I* will accept the proposal of the arbitrator *x*, $\mu_2(y)$ – probability of that the player of *II* will accept the current offer *y* and $\mu_3(z)$ – probability of that the player of *III* will accept the offer $z$ $(1-x-y)$. Owing to symmetry of game for the second and third player we believe $\mu_2(y)=\mu_3(z)$.

With probability $\mu_1\cdot\mu_2\cdot\mu_3$ on this step all players will accept the current values $(x,y,z)$ offered by the arbitrator. Then the first player will receive a piece of cake the size *x*. With opposite probability $(1-\mu_1\cdot\mu_2\cdot\mu_3)$ at least one of players will reject the piece of cake offered it. Then game will pass to the following step and players will divide the discounted cake. Thus, the optimality equation for a prize of the player *I* on *k*-step has the following appearance.

$$H_k^{(1)} = \int_0^1 dx \int_0^{1-x} \left\{\mu_1\mu_2\mu_3 \cdot x + (1-\mu_1\mu_2\mu_3)\cdot \delta H_{k-1}^{(1)}\right\} m(m+1)x^{m-1}dy \to \max_{\mu_1}$$

or

$$H_k^{(1)} = m(m+1)\left\{\int_0^1 \mu_1 x^{m-1}\left(x - \delta H_{k-1}^{(1)}\right)dx \int_0^{1-x} \mu_2\mu_3 dy\right\} + \delta H_{k-1}^{(1)} \to \max_{\mu_1} \qquad (2)$$

The purpose of the player of $I$ - to maximize the prize. In a formula (2) it can affect only value of the first integral. Therefore we will consider separately expression at $\mu_1$. Find the equilibrium among the threshold strategies. Let

$$\mu_1(x) = \mathrm{I}\{x \ge b\}, \quad \mu_2(y) = \mathrm{I}\{y \ge a\}, \quad \mu_2(1-x-y) = \mathrm{I}\{1-x-y \ge a\}$$

where $I\{A\}$ - event indicator $A$.

Thus, on this step the first player accepts the proposal of the arbitrator if it not less threshold of $b$ and the second and the third agree with proposals of the arbitrator if they are equal or more number of $a$.

Then in a formula (2) expression at $\mu_1$ will assume the following view

$$x^{m-1}\left(x - \delta H_{k-1}^{(1)}\right)\int_0^{1-x} \mu_2\mu_3 dy = x^{m-1}\left(x - \delta H_{k-1}^{(1)}\right)\int_a^{1-x-a} dy \mathrm{I}\{b \le x \le 1-2a\} + 0\cdot \mathrm{I}\{x > 1-a\} = x^{m-1}\left(x - \delta H_{k-1}^{(1)}\right)(1-x-2a)\mathrm{I}\{b \le x \le 1-2a\}$$

Thus the optimality equation for $I$ in $k$-th step is of the following form

$$H_k^{(1)} = \left(1 - 2\delta H_{k-1}^{(2)}\right)^{m+1}\left[\frac{m}{m+2}\left(1-2\delta H_{k-1}^{(2)}\right) - \delta H_{k-1}^{(1)}\right] + \left(\delta H_{k-1}^{(1)}\right)^{m+1}\left[1 - 2\delta H_{k-1}^{(2)} - \frac{m}{m+2}\delta H_{k-1}^{(1)}\right] + \delta H_{k-1}^{(1)}$$

where $a = \delta H_{k-1}^{(2)}$, $b = \delta H_{k-1}^{(1)}$.

Now consider the optimality equation for player $II$. With probability $\mu_1 \cdot \mu_2 \cdot \mu_3$ all players on this step will accept the values offered by the arbitrator. Then the second player will receive a piece of cake the size $y$. With opposite probability $(1-\mu_1 \cdot \mu_2 \cdot \mu_3)$ one, two or all three players will reject proposals of the arbitrator, then game will pass to the following step. We receive prize function in the following look

$$H_k^{(2)} = \int_0^1 dy \int_0^{1-y} \left\{\mu_1\mu_2\mu_3 \cdot y + (1-\mu_1\mu_2\mu_3)\cdot \delta H_{k-1}^{(2)}\right\} m(m+1)x^{m-1}dx \to \max_{\mu_2} \qquad (3)$$

The purpose of the player of $II$ also is maximizing the prize. In a formula (3) it can affect only value of the first integral. Therefore we will consider separately expression at $\mu_2$.

$$\left(y - \delta H_{k-1}^{(2)}\right)\int_0^{1-y} x^{m-1}\mu_1\mu_3 dx = \frac{1}{m}\left(y - \delta H_{k-1}^{(2)}\right)\left((1-y-a)^m - b^m\right)\mathrm{I}\{a \le y \le 1-a-b\} + 0\cdot \mathrm{I}\{y > 1-a-b\}$$

Finally, for player $II$ in $k$-th step we have

$$H_k^{(2)} = \frac{1}{m+2}(1-2\delta H_{k-1}^{(2)})^{m+2} - \frac{m+1}{2}(\delta H_{k-1}^{(1)})^m\left(1 - 2\delta H_{k-1}^{(2)} - \delta H_{k-1}^{(1)}\right)^2 +$$
$$\frac{-1}{m+2}(\delta H_{k-1}^{(1)})^{m+1}\left[(2+m)(1-2\delta H_{k-1}^{(2)}) - \delta H_{k-1}^{(1)}(1+m)\right] + \delta H_{k-1}^{(2)}$$

We received recurrent formulas for payoff calculation for the first player $H_k^{(1)}$ and payoff for the second and third players $H_k^{(2)} = H_k^{(3)}$. Setting various values for discounting coefficient $\delta$, numbers of $m$ and initial values $H_k$, it is possible to define optimum strategy of players and to find their prizes at the set parameters of a task.

Thus, in game of three persons of a sharing of cake in which the final decision requires a consent of all players (full consensus), optimum strategy of players on $k$-step have the following appearance

$$\mu_1(x) = \mathrm{I}\left\{x \geq \delta H_{k-1}^{(1)}\right\}, \quad \mu_2(y) = \mathrm{I}\left\{y \geq \delta H_{k-1}^{(2)}\right\}, \quad \mu_2(z) = \mathrm{I}\left\{z \geq \delta H_{k-1}^{(3)}\right\}$$

Similarly it is possible to consider a situation when the second or third player have advantages at cake division.

For example, at $m=1$ in case of lack of discounting ($\delta=1$) and infinite time of negotiations ($k \to 8$) players can wait the moment when the arbitrator will offer every the third part of cake. In this case the value of the game satisfies to recurrence relation

$$H_k = \delta H_{k-1} + \frac{1}{3}\left(1 - 3\delta H_{k-1}\right)^3$$

and optimal strategies of players in $k$-th step are

$\mu_1(x)=I\{x \geq \delta H_{k-1}\}$, $\mu_2(y)=I\{y \geq \delta H_{k-1}\}$, $\mu_3(z)=I\{z \geq \delta H_{k-1}\}$.

Let's consider a case when the first player has advantage. Let $k_1=2$, $k_2=k_3=1$. Then $f(x,y) = 6x$, where $x + y=1$ and $x, y=0$. Notice that the problem is symmetric for the players *II* and *III*. In this case the value of the game is satisfied to the recurrence relation

$$\begin{cases} H_k^{(1)} = \dfrac{\left(-1+2\delta H_{k-1}^{(2)} - \delta H_{k-1}^{(1)}\right)\left(-1+2\delta H_{k-1}^{(2)} - \delta H_{k-1}^{(1)}\right)^3}{2} + \delta H_{k-1}^{(1)} \\ H_k^{(2)} = \dfrac{\left(-1+2\delta H_{k-1}^{(2)} - 3\delta H_{k-1}^{(1)}\right)\left(-1+2\delta H_{k-1}^{(2)} - \delta H_{k-1}^{(1)}\right)^3}{4} + \delta H_{k-1}^{(2)} \end{cases}$$

Table 1:

| $k$ | $H_k$ | $\delta$=0.9 | $\delta$=0.95 | $\delta$=0.999 | $\delta$=1 |
|---|---|---|---|---|---|
| 50 | $H_k^{(1)}$ | 0.264497 | 0.282353 | 0.479883 | 0.5 |
| | $H_k^{(2)} = H_k^{(3)}$ | 0.204614 | 0.21819 | 0.22844 | 0.237297 |
| 1000 | $H_k^{(1)}$ | 0.264584 | 0.273501 | 0.374372 | 0.5 |
| | $H_k^{(2)} = H_k^{(3)}$ | 0.204745 | 0.221441 | 0.267905 | 0.240415 |

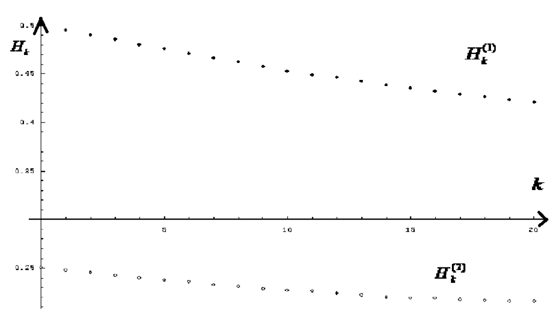


Fig. 1: The values for the 3-person game $H_k$

For $\delta=1$, as $k \to \infty$ from the system of equations we obtain a recurrent formula $H_k^{(2)} = \frac{1}{2}\left(1 - H_k^{(1)}\right)$.

In Table 1 the values of payoffs $H_k$ are presented for $k=50$ and $k=1000$ for initial values $H_0^{(1)} = 0.1$ and $H_0^{(2)} = H_0^{(3)} = 0$. We see that for $\delta=1$ and great horizon player *I* receives a half of the cake and other players obtain a quota of the cake.

On Figure 1 the graphs of the values for the 3-person game $H_k$ are presented. The graph $H_k$ corresponds to the model of full consensus. $H_k^{(1)}$ is calculated for player *I* with the parameters of the Dirichlet distribution $k_1=2$, $k_2=k_3=1$ and $H_k^{(2)}$' for player *II* and *III*. Other parameters are K=20, $\delta$=0.99, $H_0^{(1)} = 0.5$, $H_0^{(2)} = H_0^{(3)} = 0.25$.

## CONCLUSION

In economy, the problem of fair division of goods is fundamental. It is, for example, distribution of territories, material values, spheres of influence, marital property in a divorce, sovereignty in an international dispute, Social Security, etc.

This work continues the series of models of negotiations such as negotiations of the worker and employer [15, 16], exchange games [17], models with arbitration [18], the cake division, etc.

In the paper the stochastic procedure of cake division problem is proposed. This model can be adapted to different real situations. In fair case we can choose the parameters of the Dirichlet distribution be equal. If some of participants have a greater weight we can increase the parameter of distribution which corresponds her. The arbitrator is presented by a generator of random numbers and special stochastic protocol.

Analytical expression for a prize of each of three players in the form of recurrent formulas is found. The solution of a task will depend on model parameters: interval of time which has been taken away for negotiations of $k$, coefficient of discounting of cake $\delta$, the parameters of the Dirichlet distribution $m$. The optimum behavior of participants of negotiations is received in a class of threshold strategy.

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