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Solutions of Second Order Singular Boundary Value Problems

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Abstract: In this paper, we propose a simple method for solving the singular boundary value problems. The proposed method is tested on several problems and its results are very encouraging.

Key words: Singular boundary value problems · B-spline · Perturbation method · Analytic solution · Neighbourhood

INTRODUCTION

The boundary value singular differential equations occur in many disciplines e.g. Mathematics, Physics, Engineering. They are basically used to model physical phenomena in astrophysics, electro-hydrodynamics, theory of thermal explosions [1], to name a few. In this paper, we will consider the following class of singular boundary value problems.

$$y''(x)+\frac{\alpha}{x}y'(x)+y(x)=f(x)\quad 0<x\le 1 \tag{1}$$

subject to the boundary conditions

$$y'(0)=0,\ y(1)=A.$$

Where f is analytical function of x defined over [0,1], $\alpha\geq 0$ and A is a real number.

These types of problems have been investigated by several researchers [2-8]. The common approach for solving such types of problems is to modify the original problem at singular point and then an appropriate method is applied for solving the resultant boundary value problem. The basic purpose to write the differential equation in this fashion is to handle singularity. Many scientists have discussed various methods for obtaining their numerical solutions that include B-splines [1,4-7], divided difference [8-11], Homotopy method [12-17], variational iteration method [18-22]. Details about B-splines can be found in [23-25]. We write Eqn. (1) as follows.

$$xy''(x)+\alpha y'(x)+xy(x)=xf(x)\quad 0\le x\le 1 \tag{2}$$

Now this equation is defined for all values of x in [0, 1]. It is assumed that a unique solution $y(x)$ exists and is analytic in the given interval [25]. Taking $x\to 0$ gives $y'(0)=0$, which is the first boundary condition. We derive relations for obtaining higher derivatives at some point $x=x_0\in[0,1]$. Without loss of generality and because of simplicity, we can take x_0 as zero. Then using Taylor series we may write the solution $y(x)$ as follows.

$$y(x)=y(0)+\frac{1}{1!}y'(0)x+\frac{1}{2!}y''(0)x^2+\frac{1}{3!}y'''(0)x^3+\ldots+\frac{1}{n!}y^{(n)}(0)x^n+\ldots \tag{3}$$

In (3), some derivatives i.e. $y^{(i)}(0),\ i=1,2,\ldots,$ may be functions of $y(0)$. Using other boundary condition, we obtain the value of unknown $y(0)$. The method is illustrated using some examples from the literature [1].

Numerical Results: In this section, we test the efficacy of our proposed method by applying it on the same problems as discussed in [1].

Problem 1: Consider the following Bessel's equation of order zero.

$$y''(x)+\frac{1}{x}y'(x)+y(x)=0,\quad y'(0)=0,\quad y(1)=1 \tag{4}$$

Solution: We write (4) in the following form

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$$xy''(x)+y'(x)+xy(x)=0 \tag{5}$$

Taking $x\to 0$ in (5), we get the first boundary condition, i.e. $y'(0)=0$.
Differentiating (5) once, we have

$$xy^{(3)}(x)+2y''(x)+xy'(x)+y(x)=0$$

Putting $x=0$ gives

$$y''(0)=-\frac{y(0)}{2} \tag{6}$$

Differentiating (5) twice, we have

$$xy^{(4)}(x)+3y^{(3)}(x)+xy''(x)+2y'(x)=0$$

Putting $x=0$ in this equation, we get

$$3y^{(3)}(0)+2y'(0)=0 \;\; or \;\; y^{(3)}(0)=0 \tag{7}$$

Differentiating (5) thrice, we have

$$xy^{(5)}(x)+4y^{(4)}(x)+xy^{(3)}(x)+3y''(x)=0$$

Putting $x=0$ in this equation gives

$$y^{(4)}(0)=-\frac{3}{4}y''(0) \tag{8}$$

In general, the nth derivative is given by

$$y^{(n+1)}(0)=-\frac{n}{n+1}y^{(n-1)}(0)$$

It may be written as

$$y^{(2n)}(0)=\prod_{k=1}^{n}(-1)^k\frac{(2k-1)}{(2k)}y(0) \text{ and } y^{(2n-1)}(0)=0 \tag{9}$$

We take the solution in the form of Taylor series at $x=0$, i.e.

$$y(x)=y(0)+\frac{1}{1!}y'(0)x+\frac{1}{2!}y''(0)x^2+\frac{1}{3!}y'''(0)x^3+....+\frac{1}{n!}y^{(n)}(0)x^n+...$$

Putting the values of different derivatives of $y(x)$ at $x=0$ in the above eqn. and taking m terms, we have

$$y(x)=y(0)-\frac{y(0)}{2!2}x^2+\frac{3y(0)}{4!2.4}x^4-\frac{3.5y(0)}{6!2.4.6}x^6+...+(-1)^m\frac{3.5....(2m-1)y(0)}{(2m)!2.4...2m}x^{2m}$$

Simple manipulations give

$$y(x)=y(0)-\frac{y(0)}{2!2}x^2+\frac{y(0)}{(2.4)^2}x^4-\frac{y(0)}{(2.4.6)^2}x^6+...+(-1)^m\frac{y(0)}{(m!)^2 2^{2m}}x^{2m}$$

$$y(x)=y(0)\sum_{k=0}^{m}\frac{(-1)^k}{(k!)^2 2^{2k}}x^{2k} \tag{10}$$

Putting $x=1$, $y(1)=1$ in (10), we have

$$y(0)=\frac{1}{\sum_{k=0}^{m}\frac{(-1)^k}{(k!)^2 2^{2k}}} \tag{11}$$

The solution of the problem is given by (10) and $y(0)$ is given by (11). It may be noted that in the limiting case (i.e. $m\to\infty$), $y(x)$ becomes the Bessel function of order zero, *i.e.* $y(x)=J_0(x)$ and $y(0)=J_0(1)$.

Problem 2: Consider the following equation.

$$y''(x)+\frac{1}{x}y'(x)+y(x)=\frac{5}{4}+\frac{x^2}{16}, \quad y'(0)=0, \quad y(1)=\frac{17}{16} \tag{12}$$

Solution: We write the differential eqn. (12) in the following form

$$xy''(x)+y'(x)+xy(x)=\frac{5x}{4}+\frac{x^3}{16} \tag{13}$$

Performing similar steps as in Problem 1, we get

$$y''(0)=\frac{5}{8}-\frac{y(0)}{2} \tag{14}$$

$$y^{(3)}(0)=0 \tag{15}$$

$$y^{(4)}(0)=\frac{6}{64}-\frac{3y''(0)}{4} \; or \; y^{(4)}(0)=\frac{3}{8}(y(0)-1) \tag{16}$$

$$y^{(5)}(0)=0 \tag{17}$$

$$y^{(6)}(0)=-\frac{5}{6}y^{(4)}(0) \text{ or } y^{(6)}(0)=\frac{15}{48}(y(0)-1) \tag{18}$$

It may be noted that all odd derivatives at $x_0=0$ are multiple of $y'(0)$ and even derivatives are multiple of $y(0)$. We write Taylor series of the solution about $x=0$, i.e.

$$y(x) = y(0) + \frac{1}{1!}y'(0)x + \frac{1}{2!}y''(0)x^2 + \frac{1}{3!}y'''(0)x^3 + ... + \frac{1}{n!}y^{(n)}(0)x^n + ... \tag{19}$$

Taking few terms in (19) (say, 5) and noting *x=1, y(1)=17/16*, we have

$$\frac{17}{16} = y(0) + \frac{1}{2!}\left(\frac{5}{8} - \frac{y(0)}{2}\right) + \frac{1}{4!}\frac{3}{8}\left(y(0) - 1\right)$$

It gives *y(0)=1*

In this way, we have *y(0)=1, y''(0)=1/8, y* $^{(4)}(0)=0$. In fact all higher derivatives, being multiple of $y^{(4)}(0)$, are zero. Thus, we have solution from (19) as follows

$$y(x) = 1 + \frac{1}{2!8}x^2. \tag{20}$$

Problem 3: Consider the following problem

$$y''(x) + \frac{2}{x}y'(x) - 4y(x) = -2$$
$$0 < x \le 1, \quad y'(0) = 0, \; y(1) = 5.5 \tag{21}$$

Solution: We write (21) in the following form

$$xy''(x) + 2y'(x) - 4xy(x) = -2x \tag{22}$$

Putting *x=0* in (22), we get the first boundary condition. Performing similar steps as in problem 1, we have

$$y^{(2)}(0) = \frac{4y(0) - 2}{3} \tag{23}$$

$$y^{(3)}(0) = 0 \tag{24}$$

$$y^{(4)}(0) = \frac{12y''(0)}{5} \tag{25}$$

$$y^{(5)}(0) = 0 \tag{26}$$

$$y^{(6)}(0) = \frac{20y^{(4)}(0)}{7} \tag{27}$$

It may be noted that all odd derivatives are zero and even derivatives are given by the following recurrence formula

$$y^{(k)}(0) = \frac{4(k-1)y^{(k-2)}(0)}{k+1}$$

It can be written as

$$y^{(k)}(0) = \frac{4^{(k-2)/2}(4y(0)-2)}{k+1}, \text{ where } k \text{ is positive even integer} \ge 2 \tag{28}$$

Thus, the solution of the problem is given by

$$\begin{aligned} y(x) &= y(0) + \frac{y''(0)}{2!}x^2 + \frac{3.4y''(0)}{4!5}x^4 + \frac{3.4^2y''(0)}{6!7}x^6 + \frac{3.4^3y''(0)}{8!9}x^8 + ... \\ y(x) &= y(0) + \frac{3y''(0)}{8x}\left(\frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \frac{(2x)^7}{7!} + \frac{(2x)^7}{9!} + ..\right). \\ y(x) &= y(0) + \frac{3y''(0)\sinh(2x)}{8x} - \frac{3y''(0)}{4} = y(0) + \frac{(2y(0)-1)\sinh(2x)}{4x} - \frac{2y(0)-1}{2}. \\ \text{or } y(x) &= \frac{1}{2} + \frac{(2y(0)-1)\sinh(2x)}{4x} \end{aligned} \tag{29}$$

We need to find the value of *y(*0). Putting *x=1* and noting *y(1)=5.5,* in (29)*,* we have

$$\frac{11}{2} = \frac{1}{2} + \frac{(2y(0)-1)\sinh(2)}{4}$$

It gives

$$(2y(0)-1) = \frac{20}{\sinh(2)}$$

Using this value in (29), the solution is

$$y(x) == \frac{1}{2} + \frac{5\sinh(2x)}{\sinh(2)} \tag{30}$$

This is same as analytical solution of the problem.

Problem 4: Consider the following problem

$$-y''(x)-\frac{2}{x}y'(x)+(1-x^2)y(x)=x^4-2x^2+7 \qquad y'(0)=0, \quad y(1)=0 \tag{31}$$

Solution: We write (31) in the following form

$$xy''(x)+2y'(x)-x(1-x^2)y(x)=-x^5+3x^3-7x \tag{32}$$

Performing similar steps as in problem 1, we have

$$y^{(2)}(0)=\frac{y(0)-7}{3} \tag{33}$$

$$y^{(3)}(0)=0 \tag{34}$$

$$y^{(4)}(0)=\frac{3y''(0)-6y(0)+12}{5}=\frac{y(0)-7-6y(0)+12}{5}=1-y(0) \tag{35}$$

$$y^{(5)}(0)=0 \tag{36}$$

$$y^{(6)}(0)=\frac{5y^{(4)}(0)-60y^{(2)}(0)-120}{7}$$

$$y^{(6)}(0)=\frac{25(1-y(0))}{7} \tag{37}$$

It may be noted that all odd derivatives are zero and even derivatives are given by the following recurrence formula

$$y^{(k)}(0)=\frac{(k-1)y^{(k-2)}(0)-(k-1)(k-2)(k-3)y^{(k-4)}(0)}{k+1} \text{ , k is even integer } \geq 8 \tag{38}$$

It may be noted that all even derivatives except second order are multiple of *(y(0)-1)*.
The solution of the problem is given by

$$y(x)=y(0)+\frac{y(0)-7}{2!3}x^2+\frac{5(1-y(0))}{4!5}x^4+\frac{25(1-y(0))}{6!7}x^6+\frac{185(1-y(0))}{8!9}x^8+...$$
$$or \quad y(x)=y(0)+\frac{y(0)-7}{2!3}x^2+(1-y(0))\left(\frac{5}{4!5}x^4+\frac{25}{6!7}x^6+\frac{185}{8!9}x^8+..\right). \tag{39}$$

Putting *y(1)=0* in (39), we have

$$0=y(0)+\frac{y(0)-7}{2!3}+(1-y(0))\left(\frac{5}{4!5}+\frac{25}{6!7}+\frac{185}{8!9}+..\right)$$

It gives *y(0)=1*
Substituting the value of *y(0)* in (39), we have the solution

$$y(x)=1-x^2. \tag{40}$$

Problem 5: Consider the following problem

$$y''(x)+\frac{1}{x}y'(x)=\left(\frac{8}{8-x^2}\right)^2, \quad y'(0)=0, \quad y(1)=0 \tag{41}$$

Solution: We write (41) in the following form

$$x(8-x^2)^2y''(x)+(8-x^2)^2y'(x)=64x \tag{42}$$

Performing similar steps as in problem 1, we have

$$y^{(2)}(0)=\frac{1}{2} \tag{43}$$

$$y^{(3)}(0)=0 \tag{44}$$

$$y^{(4)}(0)=\frac{3}{8} \tag{45}$$

$$y^{(5)}(0)=0 \tag{46}$$

$$y^{(6)}(0)=\frac{12}{8} \tag{47}$$

It may be noted that all odd derivatives are zero and even derivatives are given by the following recurrence formula

$y^{(k)}(0)=\dfrac{(k-1)(k-2)y^{(k-2)}(0)}{8}$, k is even integer ≥ 4

Or

$$y^{(k)}(0)=\frac{(k-1)!\,y^{(2)}(0)}{8^{(k-2)/2}} \tag{48}$$

The solution of the problem is given by

$$y(x)=y(0)+\frac{y''(0)}{2}x^2+\frac{y''(0)}{4.8}x^4+\frac{y''(0)}{6.8^2}x^6+\frac{y''(0)}{8.8^3}x^8+\ldots$$

Or

$$y(x)=y(0)+\frac{y''(0)}{2}x^2+4y''(0)\left(\frac{(x^2/8)^2}{2}+\frac{(x^2/8)^3}{3}+\frac{(x^2/8)^4}{4}+..\right).$$

Or

$$y(x)=y(0)+\frac{y''(0)}{2}x^2+4y''(0)\left(-\log(1-\frac{x^2}{8})-\frac{x^2}{8}\right)$$

Or

$$y(x)=y(0)-2\log\left(1-\frac{x^2}{8}\right) \tag{49}$$

Putting *y(1)=0* in (49), we have

$$y(0)=2\log\left(\frac{7}{8}\right)$$

Thus, solution is given by

$$y(x)=2\log\left(\frac{7}{8}\right)-2\log\left(1-\frac{x^2}{8}\right)=2\log\left(\frac{7}{8-x^2}\right) \tag{40}$$

CONCLUSION

In this paper, we have discussed a very simple method that provides efficient solutions. In fact in limiting case, the solution matches the exact solution. Increasing the number of terms in the solution, numerical accuracy can be increased considerably. The method has been tested on five problems and results obtained match the exact solutions in limiting case.

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