World Applied Sciences Journal 15 (10): 1473-1483, 2011
ISSN 1818-4952

The Polynomial of Detour Index for a Graph

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Abstract: The detour index is equal to the sum of distances between all pairs of vertices of the connected graph on the longest path between corresponding vertices. We define a generating function, which we call the detour index polynomial, whose derivative is the detour index when $q = 1$. We study some of the elementary properties of this polynomial and compute it for some common graphs. Finally, we compute the detour index polynomial for some subdivision graphs with finding the detour distance in these graphs.

Key words: Detour distance · Detour index · Subdivision graph · Line graph · Total graph
2000 Mathematics subject classification: 05C12 · 05C31 and 05C38

INTRODUCTION

The detour matrix is one of the particularly important distance matrices which are based on the topological distance for vertices in a graph. It was introduces into the mathematical literature in 1969 by Frank Harary [1] and it was discussed in 1990 by Buckley and Harary [2]. The detour matrix was introduced into the chemical literature in 1994 under the name "the maximum path matrix of a molecular graph" [3-7] and theoretical graph theory contribution to finding the some interest in chemistry [8-16]. During these works, the detour index has been defined for a connected graph G as follows:

$$D(G) = \sum_{\{u,v\}\subseteq V(G)} \Delta(u,v) \qquad (1)$$

Where $\Delta(u,v)$ denotes the detour distance Which is the distance between the vertices u and v on the longest path. In [17-21], some work has been done on detour index of graphs.

In this paper, we wish to define and study the polynomial of detour index. If q is a parameter, then the detour index polynomial of G is

$$D(G;q) \sum_{\{x,y\}\in V(G)} q^{\Delta(u,v)} \qquad (2)$$

In the rest of this section, we will derive some basic properties of $D(G; q)$ and find its value when G specializes to a number of simple graphs. In section 2, the detour distances of some subdivisions graphs have been founded and in the section 3, the relations between detour index polynomials of these graphs have been concluded.

In what follows, we use $|S|$ to describe the cardinal of a set S. Also, if $f(q)$ is a polynomial in q, then deg $f(q)$ is its degree and $[q^i]\, f(q)$ is the coefficient of q^i.

The next theorem summarizes some of the properties of $D(G; q)$. Its proof follows easily from the definitions and so is omitted.

Theorem 1-1: The polynomial of detour index satisfy the following conditions:

- deg $D\,(G;q)$ = diameter of G due to the detour distance
- $[q^0]D(G;q) = 0$.

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- $$[q^1]D(G;q) = \begin{cases} |E(G)| & , \; \textit{if } G \textit{ is a acyclic graph} \\ \textit{the number of vertices with } \deg\textit{ree one} & , \quad \textit{o.w.} \end{cases}$$
- $D(G;1) = \binom{|V(G)|}{2}$
- $D'(G;1) = D(G)$. ■

Next, we find the detour index polynomial of some specific graphs. Let K_n, P_n, C_n and W_n denote the complete graph, path, cycle and wheel on n vertices, respectively. Also let $K_{m,n}$ be the complete bipartite graph on parts of size m and n. Finally, P denotes the Petersen graph and $[n] = 1 + q + q^2 + ... + q^{n-1}$. Determining the detour index polynomial of these graphs is a matter of simple counting, so the proof of the next result is also omitted.

Theorem 2-2: Some specific detour index polynomials are as follows:

- $D(K_n;q) = \binom{n}{2} q^{n-1}$.
- $D(K_{m,n};q) = mnq^{n+m-1} + \left(\binom{n}{2} + \binom{m}{2}\right) q^{n+m-2}$.
- $D(W_n;q) = \binom{n}{2} q^{n-1}$.
- $D(P;\, q) = 15q^8 + 30q^9$.
- $D(P_n;q) = \frac{q}{1-q}(n - [n])$.
- $D(C_{2n};q) = nq^{n-1}\left(2[n]-1\right)$.
- $D(C_{2n};q) = (2n+1)q^{n+1}[n]$. ■

Combining the previous theorem with number of 5 of theorem 1-1, we obtain the well-known detour indices of theses graphs.

Theorem 1-3: Some specific detour indices are as follows:

- $D(K_n) = \binom{n}{2}(n-1)$.
- $D(K_{m,n}) = mn(n+m-1) + \left(\binom{n}{2} + \binom{m}{2}\right)(n+m-2)$.
- $D(W_n) = \binom{n}{2}(n-1)$.
- $D(P) = 390$..
- $D(P_n) = \binom{n+1}{3}$.
- $D(C_{2n}) = n\left(6\binom{n}{2} + n\right)$.
- $D(C_{2n+1}) = 3\left(\frac{n(n+1)(2n+1)}{2}\right)$. ■

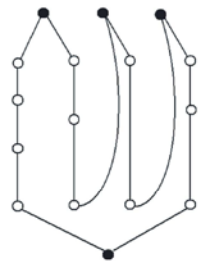

Fig. 1: The Hamiltonian cycle of $G_1 + G_2$ which $DP(G_1) = n_2 = 4$.

Now, we compute the polynomial of detour index of a graph operator for special graphs. Let $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ be two graphs with vertex sets V_i and edge sets E_i such that $|V_i| = n_i$ and $|E_i| = k_i$ for $i = .1,2$. The join of two graphs G_1 and G_2 which is shown $G_1 + G_2$, has vertex set $V(G_1 + G_2) = V_1 \cup V_2$ and edge set $V(G_1 + G_2) = E_1 \cup E_2 \cup \{uv|u \in V_1, v \in V_2\ or\ v \in V_1, u \in V_2\}$. Furthermore, we use the notation $MDP(G)$ and $DP(G)$ which are the minimum number of distinct paths of a graph G and the number of distinct paths of a graph G in following computation.

Theorem 1-4: Let $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ be two connected graphs such that $|V_1| = n_1$, $|V_2| = n_2$ and $n_1 \geq n_2$. The join graph $G_1 + G_2$ is Hamiltonian iff there exist a set of distinct paths of graph G_1 with size $DP(G_1) \leq n_2$.

Proof: Let $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ be two connected graphs such that $|V_1| = n_1$, $|V_2| = n_2$, and $n_1 \geq n_2$.
($\Rightarrow$) If the join graph $G_1 + G_2$ is Hamiltonian, we consider the segments of Hamiltonian cycle of $G_1 + G_2$ which contains just vertices of G_1 as distinct paths. Therefore, it is clear that there exist a set of distinct paths of graph G_1 with size $DP(G_1) \leq n_2$.

($\Leftarrow$) There exist a set of distinct paths of graph G_1 with size $DP(G_1) \leq n_2$. At first, suppose $DP(G_1) = n_2$. Then we consider the Hamiltonian cycle which constructed with paths and vertices of G_2 and is similar to cycle of Figure 1. Therefore it is clearly that $G_1 + G_2$ is Hamiltonian.

Now, suppose $DP(G_1) < n_2$. In this case, we divide the distinct paths G_1 to several small paths (it is possible some paths are P_1) that we have new condition $DP'(G_1) = n_2$. The equality $DP'(G_1) = n_2$ is doable because of $n_1 \geq n_2$. Hence, according to the last part of this proof, the desire result can be concluded. ■

Result 1-5: Let $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ be two connected graphs such that $|V_1| = n_1$, $|V_2| = n_2$ and $n_1 \geq n_2$. If $MDP(G_1) \leq n_2$, the join graph $G_1 + G_2$ is Hamiltonian.

Result 1-6: Let $G_1 = (V_1,E_1)$ and $G_2 = (G_2,E_2)$ be two connected graphs such that $|V_1| = n_1$, $|V_2| = n_2$ and $n_2 \geq n_2$. If there exist a set of distinct paths of graph G_1 with size $DP(G_1) \leq n_2 - 2$, then, $\Delta(u,v) = (n_1 + n_2 - 2)$ which u,v are nonadjacent in graph $G_1 + G_2$.

Proof: G_1 and G_2 be two connected graphs such that $|V_1| = n_1$, $|V_2| = n_2$, $n_1 \geq n_2$ and there exist a set of distinct paths of graph G_1 with size $DP(G_1) \leq n_2 - 2$. Suppose u,v are nonadjacent in graph $G_1 + G_2$. Therefore due to definition of $G_1 + G_2$, the vertices u,v are not adjacent in graph G_1 or G_2. If the vertices u,v are nonadjacent in graph G_1, then we divide the paths contain u *and* v to two part due to G_1 is connected. Hence we get the condition $DP'(G_1) \leq n_2$ which $DP'(G_1) = DP(G_1) + 2$. So, using the Theorem (1-4), $G_1 + G_2$ is Hamiltonian and $\Delta(u,v) = (n_1 + n_2 - 2)$.

If vertices u *and* v are nonadjacent in graph G_2, then we divide the one path to two parts and separate one vertex from another path. Therefore, there exists $DP'(G_1) \leq n_2$ paths which one path is P_1. Therefore by the procedure which is shown in Figure 1, we consider the Hamiltonian cycle of $G_1 + G_2$ such that the path P_1 is connected to u *and* v. Then, $G_1 + G_2$ is Hamiltonian and $\Delta(u,v)=(n_1 + n_2 - 2)$. ■

Using the Theorem (1-4) and Result (1-6), we state the detour index polynomial of joint graph $G_1 + G_2$ which G_1 and G_2 are connected graphs with some properties.

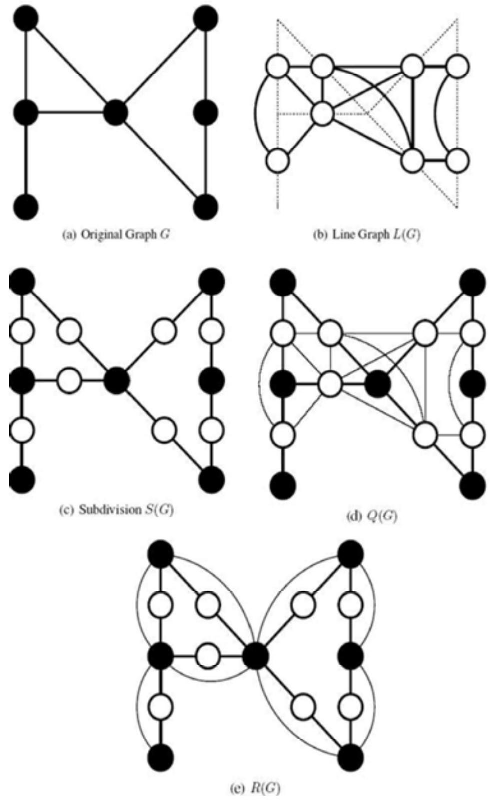

Fig. 2: The subdivision operators S, R and Q.

Theorem 1-7: Suppose G_1 and G_2 are connected and nontrivial (not equal to K_1) such that $|V_1| = n_1$, $|V_2| = n_2$ and $n_1 \geq n_2$. If there exist a set of distinct paths of graph G_1 with $DP(G_1) \leq n_2 - 2$. Then with the preceding notation.

$$D\left(G_1 + G_2; q\right) = \left(k_1 + k_2 + n_1 n_2\right) q^{n_1 + n_2 - 1} + \left(\binom{n_1}{2} - k_1 + \binom{n_2}{2} - k_2\right) q^{n_1 + n_2 - 2}$$

Proof: Due to Theorem (1-4) and Result (1-6), we can conclude the desire relation with easy summation. ■

Now, we find the detour distances in some subdivisions graphs.

Distances in Subdivisions Graphs: Firstly, we restate subdivision graphs which constructed from a graph G.

Suppose $G = (V,E)$ is a connected graph with the vertex set $V(G)$ and the edge set $E(G)$. Give an edge $e = (u,v)$, let $V(e) = \{u,v\}$. The line graph $L(G)$, the subdivision graph $S(G)$ and the total graph $T(G)$ which are related graphs to graph G have been defined as follows (See [22]):

Line graph *L(G)*: $L(G)$ is the graph whose vertices correspond to the edges of G with two vertices being adjacent if and only if the corresponding edges in G have a vertex in common. See Figure 2(b).

Subdivision Graph: $S(G)$ is the graph obtained from G by replacing each of its edge by a path of length two, or equivalently, by inserting an additional vertex into each edge of G. See Figure 2(c).

Two extra subdivision operators named $R(G)$ and $Q(G)$ are defined as follows [22]:

$R(G)$ is defined as the graph obtained from G by adding a new vertex corresponding to each edge of G and by joining each new vertex to the end vertices of the edge corresponding to it. See Figure 2(e).

$Q(G)$ is the graph obtained from G by inserting a new vertex into each edge of G and by joining edges those pairs of these new vertices which lie on adjacent edges of G. *S*ee Figure 2(d).

Given $G = (V,E)$, where $|E(G)| \subset \binom{V(G)}{2}$, we may define two other sets that we use frequently:

$EE(G) := \{\{e,e'\} |\ e,e' \in E(G), e \neq e', |V(e) \cap V(e')| = 1\}$

$EV(G) := \{\{e,v\} |\ e \in E(G), V(G) \ni v \in V(e)\}$

We can write the subdivision operators above as follows:

$L(G) := (E(G), EE(G))$
$S(G) := (V(G) \cup E(G), EV(G))$
$R(G) := (V(G) \cup E(G), E(G) \cup EV(G))$
$Q(G) := (V(G) \cup E(G), EE(G) \cup EV(G))$

Now, we find the distances between subdivision graphs.

Lemma 2-1: For each two vertices x and y, we have:

$$\Delta_G(x,y) = \frac{1}{2}\Delta_{S(G)}(x,y) = \frac{1}{2}\Delta_{R(G)}(x,y)$$

Proof: Using the definitions of subdivisions graphs $S(G)$ and $R(G)$, we can conclude the desire results easily. (Use the Figure 2). ■

To simplifying the following computations, we use the notations which defined as follows.

Definition 2-2: $M = \{\{e, f\} \subseteq E(G) | e, f \textit{ are adjacent}\}$ and we have: $|M| = |E(L(G))|$,

$N = \{\{x, y\} \subseteq V(G) | x, y \textit{ are adjacent}\}$ and we have: $|N| = |E(G)|$,

$O = \{\{x, e\} x, y \in V(G) e \in E(G) \ s.t.\ e = xy\}$ and we have: $|O| = 2|E(G)|$,

$D_1 = \{\{x, y\} \subseteq V(G) | x \textit{ or } y \textit{ has degree one}\}$ and.

$D_2 = \{\{x, y\} \subseteq V(G) | x \textit{ and } y \textit{ has degree one}\}$

Lemma 2-3: For each two edge e and f, we have:

$$\Delta_{L(G)}(e,f) = \frac{1}{2}\Delta_{S(G)}(e,f) = \begin{cases} \frac{1}{2}\Delta_{R(G)}(e,f) - 1 & ,\{e,f\} \notin M \\ \frac{1}{2}\left(\Delta_{R(G)}(e,f) - 1\right) & ,\{e,f\} \in M \end{cases}$$

Proof: The first equality is easy for computation. And using the definitions of subdivisions graphs, $S(G)$ and $R(G)$ and their figures, the distances between two adjacent and nonadjacent edges of $S(G)$ and $R(G)$ on longest path have the differences 1 or 2, respectively. Then, the result can be concluded.

Lemma 2-4: For each vertex *x* and edge *e*, we have:

$$\Delta_{S(G)}(x,e)=\begin{cases}\Delta_{R(G)}(x,e)-1 & x\notin e\\ \Delta_{R(G)}(x,e) & x\in e\end{cases}$$

Proof: Due to the definitions of subdivisions graphs, $S(G)$ and $R(G)$ and their figures, the distances between a vertex *x* and an edge *e* in $S(G)$ and $R(G)$ are equal if x is not end vertex of edge *e*. Also, if x is the end vertex of edge *e*, then the distances between a vertex *x* and an edge *e* in $S(G)$ and $R(G)$ are not equal and have the differences in 1. Therefore, the desire results follow easily. ■

Lemma 2-5: For each two vertices *x* and *y*, we have:

$$\Delta_G(x,y)=\frac{1}{2}\Delta_{S(G)}(x,y)=\begin{cases}\frac{1}{2}\Delta_{Q(G)}(x,y)-1 & , \quad \{x,y\}\notin N \ and \ \{x,y\}\notin D_1\cup D_2\\ \frac{1}{2}\left(\Delta_{Q(G)}(x,y)-1\right) & , \quad \{x,y\}\in N \ or \ \{x,y\}\in D_1\\ \frac{1}{2}\Delta_{Q(G)}(x,y) & , \quad \{x,y\}\in D_2\end{cases}$$

Proof: By using the similar procedure to above lemmas for this lemma and considering the definitions of subdivisions graphs, $S(G)$ and $Q(G)$ and their figures, we can obtain the results. ■

Lemma 2-6: For each two edge *e* and *f*, we have:

$$\Delta_{L(G)}(e,f)=\frac{1}{2}\Delta_{S(G)}(e,f)=\frac{1}{2}\Delta_{Q(G)}(e,f)$$

Proof: The results can be concluded only by considering the definitions of subdivision graphs. ■

Lemma 2-7: For each vertex *x* and edge *e*, we have:

$$\Delta_{S(G)}(x,e)=\begin{cases}\Delta_{Q(G)}(x,e)-1 & ,x\notin e\\ \Delta_{Q(G)}(x,e) & ,x\in e\end{cases}$$

Proof: Due to the definitions of subdivisions graphs, $S(G)$ and $Q(G)$ and their figures, the distances between a vertex *x* and an edge *e* in $S(G)$ and $Q(G)$ are equal if x is not end vertex of edge *e*. Also, if x is the end vertex of edge *e*, then the distances between a vertex *x* and an edge *e* in $S(G)$ and $Q(G)$ are not equal and have the differences in 1. Therefore, the desire results follow easily.

Now, we find the relation between detour indices of subdivision graphs.

Theorem 2-8: Let *G* be a graph and $S(G)$, $L(G)$, $R(G)$ and $Q(G)$ be the subdivision graphs of *G*. Therefore with the preceding notation, we have:

$$D(R(G))-D(Q(G))=2\binom{|E(G)|}{2}-|E(L(G))|-2\binom{|V(G)|}{2}+|N\cup D_1|+2|D_2|$$

Proof: Due to the Lemmas (2-1, 2-3, 2-4, 2-5, 2-6 and 2-7) and Definition (2-2), the detour indices of $R(G)$ and $Q(G)$ are

$$\begin{aligned}
D(R(G)) &= \sum_{\{x,y\}\subseteq V(G)} \Delta_{R(G)}(x,y) + \sum_{\{x,y\}\subseteq E(G)} \Delta_{R(G)}(x,y) + \sum_{\substack{x\in V(G)\\ y\in E(G)}} \Delta_{R(G)}(x,y) \\
&= \sum_{\{x,y\}\subseteq V(G)} 2\Delta_G(x,y) + \sum_{\substack{\{x,y\}\subseteq E(G)\\ and\ \{x,y\}\in M}} \left(2\Delta_{L(G)}(x,y)+2\right) + \sum_{\{x,y\}\in M} \left(2\Delta_{L(G)}(x,y)+1\right) + \\
&\quad \sum_{\substack{x\in V(G), y\in E(G)\\ \{x,y\}\notin O}} \left(\Delta_{S(G)}(x,y)+1\right) + \sum_{\substack{x\in V(G), y\in E(G)\\ \{x,y\}\in O}} \Delta_{S(G)}(x,y) \\
&= 2D(G) + 2D(L(G)) + 2\binom{|E(G)|}{2} - |E(L(G))| + \\
&\quad \sum_{\substack{x\in V(G), y\in E(G)\\ \{x,y\}\notin O}} \left(\Delta_{S(G)}(x,y)+1\right) + \sum_{\substack{x\in V(G), y\in E(G)\\ \{x,y\}\in O}} \Delta_{S(G)}(x,y)
\end{aligned}$$

and

$$\begin{aligned}
D(Q(G)) &= \sum_{\{x,y\}\subseteq V(G)} \Delta_{Q(G)}(x,y) + \sum_{\{x,y\}\subseteq E(G)} \Delta_{Q(G)}(x,y) + \sum_{\substack{x\in V(G)\\ y\in E(G)}} \Delta_{Q(G)}(x,y) \\
&= \sum_{\{x,y\}\subseteq E(G)} 2\Delta_{L(G)}(x,y) + \sum_{\substack{\{x,y\}\subseteq V(G)\\ and\ \{x,y\}\notin N\ and\\ x\ and\ y\ have\ not\ degree\ one}} \left(2\Delta_G(x,y)+2\right) + \sum_{\substack{\{x,y\}\in N\ or\\ \{x,y\}\in D_1}} \left(2\Delta_G(x,y)+1\right) + \\
&\quad \sum_{\substack{\{x,y\}\subseteq V(G)\ and\\ \{x,y\}\in D_2}} 2\Delta_G(x,y) + \sum_{\substack{x\in V(G), y\in E(G)\\ \{x,y\}\notin O}} \left(\Delta_{S(G)}(x,y)+1\right) + \sum_{\substack{x\in V(G), y\in E(G)\\ \{x,y\}\in O}} \Delta_{S(G)}(x,y) \\
&= 2D(L(G)) + 2D(G) + 2\binom{|V(G)|}{2} - |N\cup D_1| - 2|D_2| + \\
&\quad \sum_{\substack{x\in V(G), y\in E(G)\\ \{x,y\}\notin O}} \left(\Delta_{S(G)}(x,y)+1\right) + \sum_{\substack{x\in V(G), y\in E(G)\\ \{x,y\}\in O}} \Delta_{S(G)}(x,y)
\end{aligned}$$

Therefore, the quantity of $D(R(G)) - D(Q(G))$ is

$$2\binom{|E(G)|}{2} - |E(L(G))| - 2\binom{|V(G)|}{2} + |N\cup D_1| + 2|D_2|$$

In the next section, we obtain the detour index polynomial for subdivision graphs.

Detour Index Polynomial of Subdivision Graphs: By using the lemmas which concluded in past section, we state the detour index polynomial of subdivision graphs.

Theorem 3-1: Let G, be a graph and $S(G)$, $L(G)$ and $R(G)$ be the subdivision graphs of G. Therefore the detour index polynomial of $R(G)$ are:

$$D(R(G);q) = qD(S(G);q) + (1-q)D(G;q^2) + (q^2-q)D(L(G);q^2) + (q-q^2)\sum_{\{x,y\}\in M} q^{2\Delta_{L(G)}(x,y)} + (1-q)\sum_{\substack{x\in V(G),y\in E(G)\\ \{x,y\}\in O}} q^{\Delta_{S(G)}(x,y)} \quad (3)$$

Proof: Let G be a graph and $S(G)$, $L(G)$ and $R(G)$ be the subdivision graphs of G. By using the definition of $R(G)$ and Lemmas (2-1, 2-3 and 2-4), we have:

$$\begin{aligned}
D(R(G);q) &= \sum_{\{x,y\}\subseteq V(G)} q^{\Delta_{R(G)}(x,y)} + \sum_{\{x,y\}\subseteq E(G)} q^{\Delta_{R(G)}(x,y)} + \sum_{\substack{x\in V(G)\\ y\in E(G)}} q^{\Delta_{R(G)}(x,y)} \\
&= \sum_{\{x,y\}\subseteq V(G)} q^{\Delta_{S(G)}(x,y)} + \sum_{\substack{\{x,y\}\subseteq E(G)\\ and\ \{x,y\}\in M}} q^{\Delta_{S(G)}(x,y)+2} + \sum_{\{x,y\}\in M} q^{\Delta_{S(G)}(x,y)+1} + \\
&\quad \sum_{\substack{x\in V(G),y\in E(G)\\ \{x,y\}\notin O}} q^{\Delta_{S(G)}(x,y)+1} + \sum_{\substack{x\in V(G),y\in E(G)\\ \{x,y\}\in O}} q^{\Delta_{S(G)}(x,y)} \\
&= \sum_{\{x,y\}\subseteq V(G)} q^{\Delta_{S(G)}(x,y)} + q^2 \sum_{\{x,y\}\subseteq E(G)} q^{\Delta_{S(G)}(x,y)} + q\sum_{\substack{x\in V(G)\\ y\in E(G)}} q^{\Delta_{S(G)}(x,y)} + \\
&\quad (q-q^2)\sum_{\{x,y\}\in M} q^{\Delta_{S(G)}(x,y)} + (1-q)\sum_{\substack{x\in V(G),y\in E(G)\\ \{x,y\}\in O}} q^{\Delta_{S(G)}(x,y)} \\
&= qD(S(D);q) + (1-q)\sum_{\{x,y\}\subseteq V(G)} q^{2\Delta_G(x,y)} + (q^2-q)\sum_{\{x,y\}\subseteq E(G)} q^{2\Delta_{L(G)}(x,y)} + \\
&\quad (q-q^2)\sum_{\{x,y\}\in M} q^{2\Delta_{L(G)}(x,y)} + (1-q)\sum_{\substack{x\in V(G),y\in E(G)\\ \{x,y\}\in O}} q^{\Delta_{S(G)}(x,y)} \\
&= qD(S(G);q) + (1-q)D(G;q^2) + (q^2-q)D(L(G);q^2) + \\
&\quad (q-q^2)\sum_{\{x,y\}\in M} q^{2\Delta_{L(G)}(x,y)} + (1-q)\sum_{\substack{x\in V(G),y\in E(G)\\ \{x,y\}\in O}} q^{\Delta_{S(G)}(x,y)}.
\end{aligned}$$

Theorem 3-2: Let G be a graph and $S(G)$, $L(G)$ and $Q(G)$ be the subdivision graphs of G. Therefore the detour index polynomial of $Q(G)$ are:

$$D(Q(G);q) = qD(S(G);q) + \left(q^2-q\right)D(G;q^2) + \left(1-q\right)D(L(G);q^2) + (q-q^2)\sum_{\substack{\{x,y\}\in N\ or\\ \{x,y\}\in D_1}} q^{2\Delta_G(x,y)} + (1-q^2)\sum_{\substack{\{x,y\}\subseteq V(G)\ and\\ \{x,y\}\in D_2}} q^{2\Delta_G(x,y)} + (1-q)\sum_{\substack{x\in V(G),y\in E(G)\\ \{x,y\}\in O}} q^{\Delta_{S(G)}(x,y)} \quad (4)$$

Proof: Let G be a graph and $S(G)$, $L(G)$ and $Q(G)$ be the subdivision graphs of G. By using the definition of $Q(G)$ and Lemmas (2-5, 2-6 and 2-7), we have:

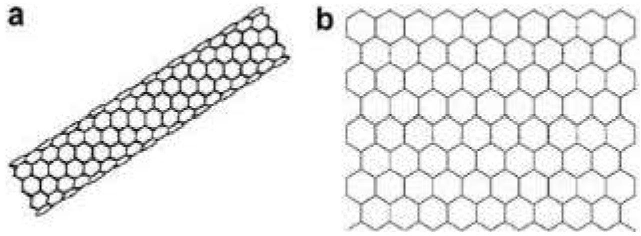

Fig. 3(a): A zigzag polyhex nanotube, (b) Its 2-dimentional lattice, p = 10 and q = 9

$$D(Q(G);q) = \sum_{\{x,y\}\subseteq V(G)} q^{\Delta_{Q(G)}(x,y)} + \sum_{\{x,y\}\subseteq E(G)} q^{\Delta_{Q(G)}(x,y)} + \sum_{\substack{x\in V(G)\\ y\in E(G)}} q^{\Delta_{Q(G)}(x,y)}$$

$$= \sum_{\{x,y\}\subseteq E(G)} q^{\Delta_{S(G)}(x,y)} + \sum_{\substack{\{x,y\}\subseteq V(G)\\ and\ \{x,y\}\notin N\ and\\ x\ and\ y\ have\ not\ degree\ one}} q^{\Delta_{S(G)}(x,y)+2} + \sum_{\substack{\{x,y\}\in N\ or\\ \{x,y\}\in D_1}} q^{\Delta_{S(G)}(x,y)+1} +$$

$$\sum_{\substack{\{x,y\}\subseteq V(G)\ and\\ \{x,y\}\in D_2}} q^{\Delta_{S(G)}(x,y)} + \sum_{\substack{x\in V(G), y\in E(G)\\ \{x,y\}\notin O}} q^{\Delta_{S(G)}(x,y)+1} + \sum_{\substack{x\in V(G), y\in E(G)\\ \{x,y\}\in O}} q^{\Delta_{S(G)}(x,y)}$$

$$= \sum_{\{x,y\}\subseteq E(G)} q^{\Delta_{S(G)}(x,y)} + q^2 \sum_{\{x,y\}\subseteq V(G)} q^{\Delta_{S(G)}(x,y)} + q \sum_{\substack{x\in V(G)\\ y\in E(G)}} q^{\Delta_{S(G)}(x,y)} +$$

$$(q-q^2) \sum_{\substack{\{x,y\}\in N\ or\\ \{x,y\}\in D_1}} q^{\Delta_{S(G)}(x,y)} + (1-q^2) \sum_{\substack{\{x,y\}\subseteq V(G)\ and\\ \{x,y\}\in D_2}} q^{\Delta_{S(G)}(x,y)} + (1-q^2) \sum_{\substack{x\in V(G), y\in E(G)\\ \{x,y\}\in O}} q^{\Delta_{S(G)}(x,y)}$$

$$= qD(S(D);q) + \left(q^2 - q\right) \sum_{\{x,y\}\subseteq V(G)} q^{2\Delta_G(x,y)} + \left(1-q\right) \sum_{\{x,y\}\subseteq E(G)} q^{2\Delta_{L(G)}(x,y)} +$$

$$(q-q^2) \sum_{\substack{\{x,y\}\in N\ or\\ \{x,y\}\in D_1}} q^{2\Delta_G(x,y)} + (1-q^2) \sum_{\substack{\{x,y\}\subseteq V(G)\ and\\ \{x,y\}\in D_2}} q^{2\Delta_G(x,y)} + (1-q) \sum_{\substack{x\in V(G), y\in E(G)\\ \{x,y\}\in O}} q^{\Delta_{S(G)}(x,y)}$$

$$= qD(S(G);q) + \left(q^2 - q\right) D(G;q^2) + \left(1-q\right) D(L(G);q^2) +$$

$$(q-q^2) \sum_{\substack{\{x,y\}\in N\ or\\ \{x,y\}\in D_1}} q^{2\Delta_G(x,y)} + (1-q^2) \sum_{\substack{\{x,y\}\subseteq V(G)\ and\\ \{x,y\}\in D_2}} q^{2\Delta_G(x,y)} + (1-q) \sum_{\substack{x\in V(G), y\in E(G)\\ \{x,y\}\in O}} q^{\Delta_{S(G)}(x,y)}.$$

Now, we find the relations between of detour index polynomial of subdivision graphs of molecular graph of zigzag polyhex nanotubes. We use the notationa p and q for the number of hexagons between two rows and number of rows, respectively. In Figure 3, you can see the molecular graph of zigzag polyhex nanotubes.

Theorem 3-3: Let G be the molecular graph of zigzag polyhex nanotube. Then the relations between detour index polynomials of subdivision graphs R(G0 and Q(G) are

$$D(R(G);q) = qD(S(G);q) + (1-q)D(G;q^2) + (q^2-q)D(L(G);q^2) +$$
$$(q-q^2)(6pq-4p)q^{2p(3q-1)} + (1-q)(6pq-2p)q^{(4pq-5)}$$

and

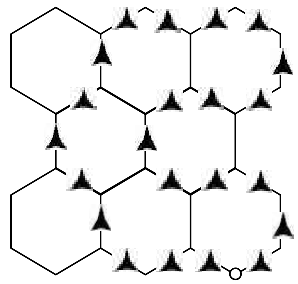

Fig. 4: The 2-dimentional lattice graph of zigzag polyhex nanotube

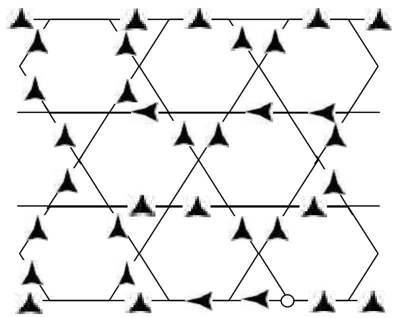

Fig. 5: The 2-dimentional lattice of line graph of zigzag polyhex nanotube

$$D(Q(G);q) = qD(S(G);q) + \left(q^2 - q\right)D(G;q^2) + (1-q)D(L(G);q^2) +$$
$$(q-q^2)(3pq-p)q^{2(2pq-3)} + (1-q)(6pq-2p)q^{(4pq-5)}.$$

Proof: Let G be the molecular graph of zigzag polyhex nanotube. The number of edges of $L(G)$ is $6pq - 4p$ and the number of vertices of $L(G)$ which is equal to the number of edges of G is $3pq - p$ and the number of vertices of G is $2pq$.

Due to Theorems (3-1 and 3-2), it is enough to compute the summations

$$\sum_{\{x,y\}\in M} q^{2\Delta_{L(G)}(x,y)}, \quad \sum_{\substack{x\in V(G), y\in E(G)\\ \{x,y\}\in O}} q^{\Delta_{S(G)}(x,y)}, \quad \sum_{\substack{\{x,y\}\in N\ or\\ \{x,y\}\in D_1}} q^{2\Delta_G(x,y)} \text{ and } \sum_{\substack{\{x,y\}\subseteq V(G)\ and\\ \{x,y\}\in D_2}} q^{2\Delta_G(x,y)}.$$

If we consider the graph G, we see that the $\Delta_G^{(x,y)}$ is equal to $2pq - 3$ (Figure 4). Therefore

$$\sum_{\substack{\{x,y\}\in N\ or\\ \{x,y\}\in D_1}} q^{2\Delta_G(x,y)} = \sum_{\substack{\{x,y\}\in N\ or\\ \{x,y\}\in D_1}} q^{2(2pq-3)} = (3pq-p)q^{2(2pq-3)}.$$

And according to Lemma (2-1), we have:

$$\sum_{\substack{x\in V(G), y\in E(G)\\ \{x,y\}\in O}} q^{\Delta_{S(G)}(x,y)} = \sum_{\substack{x\in V(G), y\in E(G)\\ \{x,y\}\in O}} q^{(4pq-5)} = (6pq-2p)q^{(4pq-5)}.$$

If we consider the graph $L(G)$, we see that $L(G)$ is Hamiltonian and $\Delta_{L(G)}(x, y)$ which x, y are vertices of $L(G)$ is equal to $3pq - p - 1$ (Figure 5). Therefore

$$\sum_{\{x,y\}\in M} q^{2\Delta_{L(G)}(x,y)} = \sum_{\{x,y\}\in M} q^{2(3pq-p)} = (6pq-4p)q^{2p(3q-1)}.$$

Also, due to the fact that G has not any vertices with degree one, therefore

$$\sum_{\substack{\{x,y\}\subseteq V(G)\ and \\ \{x,y\}\in D_2}} q^{2\Delta_G(x,y)} = 0$$

Now, we can conclude the desire results only with replacing the quantity of summations in relations 3 and 4. ¢

REFERENCES

1. Harary, F., 1969. Graph theory. Addison-Wesley, Reading, Massachusetts, pp: 203-203.
2. Buckely, F. and F. Harary, 1990. Distance in graphs. Addison-Wesley, Reading, MA, pp: 213-214.
3. Amic, D., N. Trinajstic, 1995. On the Detour Matrix. Croat. Chem. Acta, 68: 53-62.
4. Ivanciuc, O. and A.T. Balaban, 1994. Design of topological indices. Part. 8. Path matrices and derived molecular graph invariants. MATCH Commun. Math. Comput. Chem., 30: 141-152.
5. Nikolic, S., N. Trinajstic, A. Juric and Z. Mihalic, 1996. The detour matrix and the detour index of weighted graphs. Croat. Chem. Acta, 69: 1577-1591.
6. John, P.E., 1995. Ueber die Berechnung des Wiener-Index fuer ausgewaehlte Delta-dimensionale Gitterstrukturen. MATCH Commun. Math. Comput. Chem., 32: 207-219.
7 Lukovits, I., 1996. The Detour Index. Croat. Chem. Acta, 69: 873-882.
8. Trinajstic, N., S. Nikolic, B. Lucic, D. Amic and Z. Mihalic, 1997. The detour matrix in Chemistry. J. Chem. Inf. Comput. Sci., 37: 631-638.
9.. Kier, L.B. and L.H. Hall, 1990. An electrotopological state index for atoms in molecules. Pharm. Res., 7: 801-807.
10. Tratch, S.S., M.I. Stankevitch and N.S. Zefirov, 1990. Combinatorial models and algorithms in chemistry. The expanded Wiener number - a novel topological index. J. Comput. Chem., 11: 899-908.
11. Randic, M., 1994. Hosoya Matrix-A Source of New Molecular Descriptors. Croat. Chem. Acta, 67: 415-429.
12. Randic, M., 1993. Novel Molecular Descriptor for Structure-Property Studies. Chem. Phys. Lett., 211: 478-483.
13. Randic, M., X. Guo, T. Oxley and H. Krishnapriyan, 1993. Wiener matrix: Source of novel graph invariant. J. Chem. Inf. Comput. Sci., 33: 709-716.
14. Randic, M., 1995. Restricted random walks on graphs. Theor. Chem. Acta, 92: 97-106.
15. Randic, M., A.F. Kleiner and L.M. DeAlba, 1994. Distance Matrices. J. Chem. Inf. Comput. Sci., 34: 277-286.
16. Klein, D.J. and M. Randic, 1993. Resistance distance. J. Math. Chem., 12: 81-95.
17. Randic, M., D. Plavsic and M. Razinger, 1997. Double invariants. MATCH Commun. Math. Comput. Chem., 35: 243-259.
18. Mercader, A., Eduardo A. Castro and Aandrey A. Toropov, 2001. Maximum Topological Distances Based Indices as Molecular Descriptors for QSPR. 4. Modeling the Enthalpy of Formation of Hydrocarbons from Elements. Int. J. Mol. Sci., 2: 121-132.
19. Diudea, Mircea V., Gabriel Katona, Istvan Lukovits and Nenad Trinajstic, 1998. Detour and cluj-detour indices. Croat. Chem. Acta, 71: 459-471.
20. Diudea, Mircea V. and Ante Graovac, 2001. Generation and graph-theoretical properties of C_4-. Tori. MATCH Commun. Math. Comput. Chem., 44: 93-102.
21. Iranmanesh, A. and Y. Pakravesh, 2007. Detour index of $TUC_4C_8(S)$ nanotube. Ars Combinatoria, 84: 274-254.
22. Yan, W., B.Y. Yang and Y.N. Yeh, 2007. Wiener indices and polynomials of five graph operators. Applied Mathematics Letters, 20(3): 290-295.