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Entire Blow-Up Boundary Solutions to Quasilinear Elliptic Systems

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Abstract: In this paper we consider the elliptic system $\Delta_I u = p(|x|)v^{\alpha(I-1)}$, $\Delta_I v = p(|x|)u^{\beta(I-1)}$ on $R^n (n \geq 3)$ which satisfies $\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = \infty$. The parameters α and β are positive, $\Delta_I u = div(|\nabla u|^{I-2}\nabla u), I \geq 2$ and p,q are continuous functions and $\min\{p(r),q(r)\}$ does not have compact support. We show that if $\alpha, \beta \leq 1$, then such a solution exist if and only if the functions p,q satisfy

$$\int_0^\infty tp(t)(t^{\frac{2-n}{I-1}}\int_0^t s^{\frac{n-3}{I-1}}Q(s)ds)^{\alpha(I-1)}dt = \infty,$$

$$\int_0^\infty tq(t)(t^{\frac{2-n}{I-1}}\int_0^t s^{\frac{n-3}{I-1}}P(s)ds)^{\beta(I-1)}dt = \infty.$$

Where $P(r) = \int_0^r \tau p(\tau)d\tau$, $Q(r) = \int_0^r \tau q(\tau)d\tau$. For $\alpha\ \beta > 1$, we show that a solution exists if either of the above conditions faills to hold; one of the integrals is finite.

Mathematics Subject Classification: 54H25, 47H10, 54E40, 54E35
Key words: Elliptic system · Quasilinear system · Blow up solutio

INTRODUCTION

We consider the elliptic systems

$$\Delta_I u = p(|x|)v^{\alpha(I-1)},$$

$$\Delta_I v = p(|x|)u^{\beta(I-1)}, \quad x \in R^n (n \geq 3) \tag{1}$$

Where α and β are positive constants, the nonnegative functions p and q are continuous on R^n and $m(r) \equiv min\{p(r), q(r)\}$ does not have compact support. We give condition on the functions p,q,α and β which ensure the existence of a positive entire large radial solution of (1); i.e., a positive spherically symmetric solution (u,v) of (1) on R^n that satisfies

$$lim_{|x|\to\infty} u(x) = lim_{|x|\to\infty} v(x) = \infty. \tag{2}$$

This problem appears in the study of non-newtonian fluids [1] and non-newtonian filtration [2]. Such problems also arise in the study of the subsonic motion of a gas, the electric potential in some bodies [3] and Riemannian geometry [4].

Preliminary Notes: Theorem 1. If $\alpha\beta(I-1) > 1$, then the system (1) has a positive entire large solution if the p,q satisfy either the.

$$\int_0^\infty tp(t)(t^{\frac{2-n}{I-1}}\int_0^t s^{\frac{n-3}{I-1}}Q(s)ds)^{\alpha(I-1)}dt < \infty, \tag{3}$$

$$\int_0^\infty tq(t)(t^{\frac{2-n}{I-1}}\int_0^t s^{\frac{n-3}{I-1}}P(s)ds)^{\beta(I-1)}dt < \infty. \tag{4}$$

Where $P(r) = \int_0^r \tau p(\tau)d\tau$, $Q(r) = \int_0^r \tau q(\tau)d\tau$.

Proof: Let r_k be any increasing sequence of positive numbers diverging to infinity for which $m(r_k) > 0$, which exists since the function m is nonnegative and does not have compact support. Now suppose, without loss of

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generality, that (3) holds. (If, instead, inequality (4) holds, the proof is quit similar so we omit it.) Let (u_k, n_k) be a large solution to (1), established [3], on $|x| > r_k$ for each $k \in N$ and given as solution to the system ($\alpha_k > 0$)

$$u_k(r) = a_k + \int_0^r t^{\frac{1-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} p(s) v_k^{\alpha(I-1)}(s) ds dt,$$

$$v_k(r) = a_k + \int_0^r t^{\frac{1-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} q(s) v_k^{\beta(I-1)}(s) ds dt. \quad (5)$$

We show first that this sequence is monotonically decreasing on $[0, r_k]$;

$$u_{k+1}(r) > u_k(r),\ v_{k+1}(r) < v_k(r) \quad (6)$$

for all $r \in [0, r_k]$.

We will then show that the limit (u,v) is large. To this end, we show that $(u_2,v_2) < (u_1,v_1)$ on $[0,r_1]$; it will then be clear that a very similar proof works to show that $(u_{k+1},v_{k+1}) < (u_k,v_k)$ on $[0,r_k]$. We note that it is obvious that $\alpha_1 \neq \alpha_2$ for otherwise $(u_2, v_2) = (u_1,v_1)$ on $[0,r_k]$, which is impossible since (u_1,v_1) blows up at r_1 and (u_2,v_2) does not. Thus suppose $\alpha_1 < \alpha_2$. Let $R = sup(S)$ where $S = \{\rho \in [0,r_1]: (u_1(r), v_1(r)) < (u_2(r)), v_2(r)); r \in [0,\rho)\}$.

The set S is nonempty since $0 \in S$ and thus by continuity $R > 0$. If $R = r_1$, then we have a contradiction since that would mean that $lim_{r \to r_1} v_2(r) = \infty$ which cannot occur since (u_2, v_2) is continuous on $[0, r_2)$. So, suppose $R. < r_1$. Then

$$v_1(R) = a_1 + \int_0^R t^{\frac{1-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} p(s) u_1^{\beta(I-1)} ds dt$$

$$\leq a_1 + \int_0^R t^{\frac{1-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} p(s) u_2^{\beta(I-1)} ds dt$$

$$< a_2 + \int_0^R t^{\frac{1-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} p(s) u_2^{\beta(I-1)} ds dt = v_2(R).$$

Thus $v_1 < v_2$ on $[0, R]$. Similarly, we can get $u_1 < u_2$ on $[0, R]$. Thus there exists $\varepsilon > 0$ so that $(u_1,v_1) < (u_2, v_2)$ on $[0, R + \varepsilon]$ which contradicts the definition of R. Hence we must have $(u_2,v_2) < (u_1,v_1)$ on $[0, r_1]$. A similar proof produces $(u_{k+1}, v_{k+1}) < (u_k, v_k)$ on $[0,r_k]$ for all $k \in N$. It is possible that $\alpha = 0$. We need to show that this cannot has a limit (u,v) on R^n. We need to show that this cannot be the case and in fact, that (u,v) is a positive large solution. To this end, let z be a nonnegative entire large solution of $\Delta_I z = p(r)(1+H(r))^{\alpha(I-1)} dr < \infty$. it is easy to see

$$v_K(r) = a_k + \int_0^r t^{\frac{1-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} q(s) u_k^{\beta(I-1)}(s) ds dt,$$

$$\leq u_k(r) + (\int_0^r t^{\frac{1-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} q(s) ds dt) u_k^{\beta(I-1)}(r),$$

$$= u_k + H(r) u_k^{\beta(I-1)}(r),$$

Using this in the equation $\Delta_I u_k(r) = p(r) v_k^{\alpha(I-1)}$, we get

$$\Delta_I u_k(r) = p(r) v_k^{\alpha(I-1)} \leq p(r)(u_k(r) + H(r) u_k^{\beta(I-1)(r)})^{\alpha(I-1)}. \quad (7)$$

Since u_k is a positive large solution on $[0, r_k]$, it is easy to show, using a maximum principle argument, that $z \leq u_k$ on $[0, r_k]$, which yields $z \leq u$. Since u is large, there exist $r_1 > 0$ such that $u(r_1) > 0$ and

$$v(r) \geq \int_{r_1}^r t^{\frac{1-n}{I-1}} \int_{r_1}^t s^{\frac{n-1}{I-1}} q(s) u^{\beta(I-1)}(s) ds dt \geq u^{\beta(I-1)}(r_1)$$

$$\int_{r_1}^r t^{\frac{1-n}{I-1}} \int_{r_1}^t s^{\frac{n-1}{I-1}} q(s) ds dt \to \infty$$

as $r \to \infty$ This completes the proof of theorem 1.

Theorem 2: If $\alpha\beta \leq 1$, then the system (1) has a positive entire large solution if and only if the functions p and q satisfy both

$$\int_0^\infty t p(t) (t^{\frac{2-n}{I-1}} \int_0^t s^{\frac{n-3}{I-1}} Q(s) ds)^{\frac{\alpha}{I-1}} dt = \infty, \quad (8)$$

$$\int_0^\infty t q(t) (t^{\frac{2-n}{I-1}} \int_0^t s^{\frac{n-3}{I-1}} P(s) ds)^{\frac{\beta}{I-1}} dt = \infty. \quad (9)$$

Proof: Since $\alpha\beta \leq 1$, we must have either $\alpha \leq 1$ or $\beta \leq 1$. We consider only the case $\beta \leq 1$ since the proof for $\alpha \leq 1$ would be very similar. We generate a positive monotonically increasing sequence (u_k, v_k). To do this, we fix $R > 0$ and show that this sequence is bounded above on $[0, R]$. Note that $\alpha \leq v_k$ so that the $u_k(r) = a + \int_0^r t^{\frac{1-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} p(s) v_{k-1}^{\alpha(I-1)}(s) ds dt$, yields

$$u_k(r) = a + \int_0^r t^{\frac{1-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} p(s) v_{k-1}^{\alpha(I-1)}(s) ds dt,$$

$$\leq v_k(r) + \int_0^r t^{\frac{1-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} p(s) v_k^{\alpha(I-1)}(s) ds dt,$$

$$\leq v_k(r) + G(r)v_k^{\alpha(I-1)}(r),$$

Which, when substituted into the $v_k(r) = a + \int_0^r t^{\frac{1-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} q(s) u_k^{\beta(I-1)}(s) ds dt,$ and applying elementary estimates, gives

$$v_k(r) \leq a + \int_0^r t^{\frac{1-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} q(s)[v_k(s) + G(s)v_k^{\alpha(I-1)}(s)]^{\beta(I-1)} ds dt,$$

$$\leq a + \int_0^r t^{\frac{1-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} q(s)[v_k^{\beta(I-1)}(s) + G^{\beta}(s)v_k^{\alpha\beta(I-1)}(s)] ds dt,$$

$$\leq a + \int_0^r tq(t)[1 + G^{\beta(I-1)}(t)]v_k(t) dt.$$

Thus the sequence v_k is bounded on $[0,R]$ and hence convergence to a function v. Since R was arbitrary, we get the convergence of v_k to v for all $r \geq 0$. We now assume that p, q satisfy (8) and(9) and show that the solution (u, v) is large. Since $u(r) \to \infty$ then

$$u(r) \geq a + a^{\alpha(I-1)} \int_0^r t^{\frac{1-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} p(s) ds dt \to \infty.$$

To prove that $lim_{r\to\infty v(r)} = \infty$, we use the $u(r) = a + \int_0^r t^{\frac{1-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} p(s) v^{\alpha(I-1)}(s) ds dt,$ to get

$u(r) \geq a + \int_0^r t^{\frac{1-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} p(s) v^{\alpha(I-1)}(s) ds dt$ and substitute this into the

$$v(r) \geq a + \int_0^r t^{\frac{1-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} q(s) (\int_0^s \xi^{\frac{1-n}{I-1}} \int_0^\xi \tau^{\frac{n-1}{I-1}} p(\tau) v^{\alpha(I-1)} d\tau d\xi)^{\beta(I-1)} ds dt,$$

$$\geq a + a^{\alpha\beta(I-1)} \int_0^r t^{\frac{1-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} q(s) (\int_0^s \xi^{\frac{1-n}{I-1}} \int_0^\xi \tau^{\frac{n-1}{I-1}} p(\tau) v^{\alpha(I-1)} d\tau d\xi)^{\beta(I-1)} ds dt.$$

Integration by parts and elementary estimates yield

$$v(r) \geq a + \frac{a^{\alpha\beta(I-1)}}{(2-n)I} \int_0^r \frac{d}{dt} t^{\frac{2-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} q(s) G^{\beta(I-1)}(s) ds dt$$

$$\geq a + \frac{a^{\alpha\beta(I-1)}}{(n-2)(I)} (1 - \frac{1}{2^{n-2}}) \int_0^{\frac{r}{2}} tq(t) G^{\beta(I-1)}.$$

From (9) and the definition of G, $v(r) \to \infty$ as $r \to \infty$. Now to prove the converse, suppose (u,v) is a positive entire solution of (1) and at least one of the inequalities (8) or (9) does not hold. We assume (8) does not hold; i.e., p,q is a positive entire of (1), we know they satisfy a system.

$$u(r) = u(0) + \int_0^r t^{\frac{1-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} p(s) v^{\alpha(I-1)}(s) ds dt,$$

$$v(r) = v(0) + \int_0^r t^{\frac{1-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} q(s) u^{\beta(I-1)}(s) ds dt. \quad (10)$$

The first of these gives

$$u(r) \leq u(0) + G(r)v^{\alpha(I-1)},$$

Which, when substituted into the second equation yields

$$v(r) \leq v(0) + \int_0^r t^{\frac{1-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} q(s) u[u(0) + G(s)v^{\alpha(I-1)}(s)]^{\beta(I-1)} ds dt.$$

$$\leq v(0) + 2^{\beta I} \int_0^r t^{\frac{1-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} q(s)[u^{\beta I}(0) + G^{\beta I}(s)v^{\alpha\beta(I-1)}(s)] ds dt.$$

We integration by parts in the last expression to get

$$v(r) \leq v(0) + 2^{\beta I} \int_0^r t^{\frac{1-n}{I-1}} \int_0^t s^{\frac{n-1}{I-1}} q(s)[u^{\beta I}(0) + G^{\beta I}(s)v^{\alpha\beta(I-1)}(s)] ds dt.$$

$$\leq v(0) + 2^{\beta I} \int_0^r tq(t)[u^{\beta I}(0) + G^{\beta I}(t)] dt.$$

Since v is increasing and $\alpha\beta(I-1) \leq 1$, we know $v(0) \leq v(r)$ and hence $(v(0))^{\alpha\beta(I-1)} \geq v^{\alpha\beta I}(r)$. Using this in the previous inequality provides

$$v(r) \leq v(0) + 2^{\beta I} \int_0^r tq(t)[u^{\beta I}(0) + G^{\beta I}(t)v^{\alpha\beta(I-1)}(0)v(t)] dt,$$

$$\leq v(0) + 2^{\beta I} u^{\beta I}(0) \int_0^r tq(t) dt + 2^{\beta I} v^{\alpha\beta(I-1)}(0) \int_0^r tq(t) G^{\beta I}(t) v(t) dt$$

$$\leq v(0) 2^{\beta I} u^{\beta I}(0) \int_0^\infty tq(t) dt + 2^{\beta I} v^{\alpha\beta(I-1)}(0) \int_0^r tq(t) G^{\beta I}(t) v(t) dt.$$

Hence we get v bounded and therefore (u,v) cannot be large. This completes the proof of Theorem 2.

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