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On Special Smarandache TN Curves of Timelike Biharmonic Curves in the Lorentzian Heisenberg Group Heis^3

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Abstract: In this paper, we study timelike biharmonic curve in the Lorentzian Heisenberg group Heis^3. We define a special case of such curves and call it Smarandache **TN** curves in the Lorentzian Heisenberg group Heis^3. We construct parametric equations of Smarandache **TN** curves in terms of timelike biharmonic curves in the Lorentzian Heisenberg group Heis^3.

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INTRODUCTION

In a different setting, Chen [1] defined biharmonic submanifolds $M \subset E^n$ of the Euclidean space as those with harmonic mean curvature vector field, that is $\Delta H = 0$; where is the rough Laplacian and stated the following

Conjecture: Any biharmonic submanifold of the Euclidean space is harmonic, that is minimal.

If the definition of biharmonic maps is applied to Riemannian immersions into Euclidean space, the notion of Chen's biharmonic submanifold is obtained, so the two definitions agree.

The non-existence theorems for the case of non-positive sectional curvature codomains, as well as the

Generalized Chen's Conjecture: Biharmonic submanifolds of a manifold N with $Riem^N \leq 0$ are minimal, encouraged the study of proper biharmonic submanifolds, that is submanifolds such that the inclusion map is a biharmonic map, in spheres or another non-negatively curved spaces [1-3].

There are a few results on biharmonic curves in arbitrary Riemannian manifolds. The biharmonic curves are investigated in [4-12] by Körpinar *et al.*

Eells and Sampson [2] proposed an infinite-dimensional Morse theory on the manifold of smooth maps between Riemannian manifolds. Though their main results concern the Dirichlet energy, they also suggested other functionals. The interest encountered by harmonic maps and to a lesser extent by p-harmonic maps, has overshadowed the study of other possibilities, e.g. exponential harmonicity. While the examples mentioned so far are all of first-order functionals, one can investigate problems involving higher-order functionals. A prime example of these is the bienergy, not only for the role it plays in elasticity and hydrodynamics, but also because it can be seen as the next stage of investigation, should the theory of harmonic maps fail.

A smooth map $\phi : N \to M$ is said to be biharmonic ifit is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathrm{T}(\phi)|^2 \, dv_h,$$

Where $\mathrm{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ The Euler--Lagrange equation of the bienergy is given by $\mathrm{T}_2(\phi) = 0$. Here the section $\mathrm{T}_2(\phi)$ is defined by

$$\mathrm{T}_2(\phi) = -\Delta_{\phi}\mathrm{T}(\phi) + \mathrm{tr}R(\mathrm{T}(\phi), d\phi)d\phi, \qquad (1.1)$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps [13].

In this paper, we study timelike biharmonic curve in the Lorentzian Heisenberg group Heis^3. We define a special case of such curves and call it Smarandache **TN** curves in the Lorentzian Heisenberg group Heis^3. We construct parametric equations of Smarandache **TN** curves in terms of timelike biharmonic curves in the Lorentzian Heisenberg group Heis^3.

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The Lorentzian Heisenberg Group Heis^3: The Lorentzian Heisenberg group Heis^3 can be seen as the space R^3 endowed with the following multiplication:

$$(\bar{x},\bar{y},\bar{z})(x,y,z) = (\bar{x}+x, \bar{y}+y, \bar{z}+z-\bar{x}y+x\bar{y}).$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.
The Lorentz metric g is given by

$$g = -dx^2 + dy^2 + (xdy + dz)^2$$

The Lie algebra of Heis^3 has an orthonormal basis

$$\mathbf{e}_1 = \frac{\partial}{\partial z}, \mathbf{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \mathbf{e}_3 = \frac{\partial}{\partial x} \tag{2.1}$$

for which we have the Lie products

$$[\mathbf{e}_2,\mathbf{e}_3] = 2\mathbf{e}_1, [\mathbf{e}_3,\mathbf{e}_1] = 0, [\mathbf{e}_2,\mathbf{e}_1] = 0$$

with

$$g(e_1,e_1) = g(e_2,e_2) = 1,\ g(e_3,e_3) = -1.$$

Proposition: *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above, the following is true:*

$$\nabla = \begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix}, \tag{2.2}$$

where the (i,j)*-element in the table above equals* $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ *for our basis*

$$\{e_k, k = 1,2,3\} = (e_1,e_2,e_3\}$$

Moreover we put

$$R_{abcd} = R\,(e_a,e_b,e_c,e_d)$$

Where the indices a,b,c and d take the values 1,2 and 3.

$$R_{1212} = -1, R_{1313} = 1, R_{2323} = -3 \tag{2.3}$$

Timelike Biharmonic Curves in the Lorentzian Heisenberg Group Heis^3: Let $\gamma: I \to$ Heis^3 be a timelike curve on the Lorentzian Heisenberg group Heis^3 parametrized by arc length. Let $\{\mathbf{T,N,B}\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group Heis^3 along γ defined as follows:

T is the unit vector field γ' tangent to γ, **N** is the unit vector field in the direction of $\nabla_T T$ (normal to γ) and **B** is chosen so that $\{\mathbf{T,N,B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= \kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{3.1}$$

where k is the curvature of γ and τ is its torsion, [14,15].
With respect to the orthonormal basis $\{e_1,e_2e_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T}\times\mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3 \end{aligned} \tag{3.2}$$

Lemma: *(see [9]) Let* $\gamma : I$ *Heis*^{3} *be a non-geodesic timelike curve on the Lorentzian Heisenberg group Heis*^{3} parametrized by arc length. γ is biharmonic if and only if

$$\begin{aligned} &\kappa = \text{constant} \neq 0, \\ &\tau = \text{constant}, \\ &N_1B_1 = 0, \\ &\kappa^2 - \tau^2 = -1 + 4B_1^2.. \end{aligned} \tag{3.3}$$

Theorem: *(see [9]) Let* $\gamma : I \to$ *Heis*^{3} *be a non-geodesic timelike biharmonic curve on the Lorentzian Heisenberg group Heis*^{3} *parametrized by arc length. If* $N_1 = 0$ then

$$\mathbf{T}(s) = \sinh\amalg\,\mathbf{e}_1 + \cosh\amalg\sinh\psi(s)\mathbf{e}_2 + \cosh\amalg\cosh\psi(s)\mathbf{e}_3, \tag{3.4}$$

where $\amalg \in$ R.

Smarandache Tn Curves in Terms of Timelike Biharmonic Curves in the Lorentzian Heisenberg Group Heis^3:

Definition: *Let* $\gamma : I \to Heis^3$ be a unit speed regular timelike curve in the Lorentzian Heisenberg group Heis^3, *whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve.*

Now, Let Us Define a Special Form of Definition

Definition: *Let* $\gamma : I \to Heis^3$ *be a unit speed regular timelike curve in the Lorentzian Heisenberg group Heis*^{3} *and* $\{\mathbf{T,N,B}\}$ *be its moving Frenet-Serret frame.*

Smarandache **TN** *curves are defined by*

$$\wp = \frac{1}{\tau}(\mathbf{T} + \mathbf{N}). \tag{4.1}$$

Theorem: *Let $\gamma : I \to Heis^3$ be a unit speed timelike biharmonic curve and $\wp$ its Smarandache* **TN** *curve on $Heis^3$. Then, the parametric equations of $\wp$ are*

$$\begin{aligned}
x\ (s) &= \frac{1}{\tau}[\cosh\coprod\cosh[\Re s+\varepsilon]+\frac{1}{\kappa}\cosh\coprod(\Re\sinh[\Re s+\varepsilon]+2\sinh\coprod\sinh[\Re s+\varepsilon])],\\
y\ (s) &= \frac{1}{\tau}[\cosh\coprod\sinh[\Re s+\varepsilon]+\frac{1}{\kappa}\cosh\coprod(\Re\cosh[\Re s+\varepsilon]+2\sinh\coprod\cosh[\Re s+\varepsilon])],\\
z\ (s) &= \frac{1}{\tau}[\sinh\coprod-\left(\frac{1}{\Re}\cosh\coprod\sinh[\Re s+\varepsilon]+\varepsilon_1\right)\cosh\coprod\sinh[\Re s+\varepsilon]+\\
&\frac{1}{\kappa}\cosh\coprod(\Re\cosh[\Re s+\varepsilon]+2\sinh\coprod\cosh[\Re s+\varepsilon])\\
&.(\frac{\kappa}{\Re^2}\cosh\coprod\sinh[\Re s+\varepsilon]+\frac{2\kappa}{\Re^2}\sinh\coprod\cosh\coprod\sinh[\Re s+\varepsilon]+\varepsilon_2 s+\varepsilon_3)],
\end{aligned}\tag{4.2}$$

where ε, ε_1, ε_2, ε_3 are constants of integration and $\Re=(\pm\frac{\kappa}{\cosh\coprod}-2\sinh\coprod)$.

Proof. Since $|\nabla_T T| = k$ we obtain

$$\psi(s)=\frac{\kappa-\sinh 2\coprod}{\cosh\coprod}s+\varepsilon,\tag{4.3}$$

where $\varepsilon \in R$.

Thus (3.4) and (4.3), we have

$$\mathbf{T}=\sinh\coprod e_1+\cosh\coprod\sinh\left[\Re s+\varepsilon\right]\mathbf{e}_2+\cosh\coprod\cosh\left[\Re s+\varepsilon\right]\mathbf{e}_3,\tag{4.4}$$

where $\Re=\frac{\kappa-\sinh 2\coprod}{\cosh\coprod}$.

Using (2.1) in (4.4), we obtain

$$\mathbf{T}=(\cosh\coprod\cosh[\Re s+\varepsilon],\cosh\coprod\sinh[\Re s+\varepsilon],\sinh\coprod - x(s)\cosh\coprod\sinh[\Re s+\varepsilon]).$$

From (2.1), we get

$$\mathbf{T}=(\cosh\coprod\cosh[\Re s+\varepsilon],\cosh\coprod\sinh[\Re s+\varepsilon],\sinh\coprod-\left(\frac{1}{\Re}\cosh\coprod\sinh[\Re s+\varepsilon]+\varepsilon_1\right)\cosh\coprod\sinh[\Re s+\varepsilon]),\tag{4.5}$$

where ε_1 is constant of integration.

Using (3.1) and (4.4), we get

$$\nabla_{\mathbf{T}}\mathbf{T}=\cosh\coprod(\Re\cosh[\Re s+\varepsilon]+2\sinh\coprod\cosh[\Re s+\varepsilon])\mathbf{e}_2+(\Re\sinh[\Re s+\varepsilon]+2\sinh\coprod\sinh[\Re s+\varepsilon])]\mathbf{e}_3.$$

By the use of Frenet formulas, we get

$$\mathbf{N}=\frac{1}{\kappa}\nabla_{\mathbf{T}}\mathbf{T}=\frac{1}{\kappa}\cosh\coprod(\Re\cosh[\Re s+\varepsilon]+2\sinh\coprod\cosh[\Re s+\varepsilon])\mathbf{e}_2+(\Re\sinh[\Re s+\varepsilon]+2\sinh\coprod\sinh[\Re s+\varepsilon])\mathbf{e}_3].\tag{4.6}$$

Substituting (2.1) in (4.6), we have

$$\begin{aligned}
\mathbf{N}=\frac{1}{\kappa}\cosh\coprod((\Re\sinh[\Re s+\varepsilon]+2\sinh\coprod\sinh[\Re s+\varepsilon]),C(\Re\cosh[\Re s+\varepsilon]+2\sinh\coprod\cosh[\Re s+\varepsilon]),\\
(\Re\cosh[\Re s+\varepsilon]+2\sinh\coprod\cosh[\Re s+\varepsilon]).(\frac{\kappa}{\Re^2}\cosh\coprod\sinh[\Re s+\varepsilon]+\frac{2\kappa}{\Re^2}\sinh\coprod\cosh\coprod\sinh[\Re s+\varepsilon]+\varepsilon_2 s+\varepsilon_3)),
\end{aligned}\tag{4.7}$$

where ε_2, ε_3 are constants of integration.

Finally, we substitute (4.5) and (4.7) into (4.1), we get (4.2). The proof is completed.

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