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On Bounded Boundary and Bounded Radius Rotation Related with Janowski Function

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Abstract: The aim of this paper is to study the subclasses of Janowski functions with bounded boundary and bounded radius rotations of order a. The order of a function from the class of Janowski functions with bounded boundary rotations to be from bounded radius rotations is of major interest and some of its applications are also discussed here.

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INTRODUCTION

Let *A* be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

analytic in the open unit disk E = {z: |z|<1} If *f* (z) and g (z) are analytic in *E,* we say that *f* (z) is subordinate to g (z), written $f \prec g$ or $f(z) \prec g(z)$ if there exists a Schwarz function w (z) in *E* such that *f* (z) = g (w (z)).
For any two analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and

$$g(z) = \sum_{n=0}^{\infty} a_n z^n \quad (z \in E)$$

the convolution (Hadamard product) of *f* (z) and g (z) is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \quad (z \in E)$$

We denote by $S^*(\alpha)$, $C(\alpha)$, $(0 \le \alpha < 1)$, the classes of starlike and convex functions of order α respectively and are defined as:

$$S^*(\alpha) = \left\{ f \in A : \mathrm{Re}\frac{zf'(z)}{f(z)} > \alpha,\ z \in E \right\}$$

$$C(\alpha) = \left\{ f \in A : zf'(z) \in S^*(\alpha),\ z \in E \right\}$$

For $\alpha = 0$, we have the well known classes of starlike and convex univalent functions denoted by S^* and C respectively.

Let P[A,B] be the class of functions *h*, analytic in *E* with h (0) = 1 and

$$h(z) \prec \frac{1+Az}{1+Bz},\ -1 \le B < A \le 1.$$

This class was introduced by Janowski [2]. The class P[A, B] is connected with the class *P* of functions with positive real parts by the relation

$$h \in P[A,B] \Leftrightarrow \frac{(B-1)h - (A-1)}{(B+1)h - (A+1)} \in P \qquad (1.1)$$

Later on, Sokól [13] studied this class of Janowski functions with $A \in \mathbb{C}$ and $B \in [-1,0]$, for detail, see [3,11].
Let $P[A,B,\alpha]$ be the class of functions p_1, analytic in *E* with $p_1(0) = 1$ and

$$p_1(z) \prec \frac{1 + [(1-\alpha)A + \alpha B]z}{1+Bz},\ A \in \mathbb{C},\ B \in [-1,0],\ 0 \le \alpha < 1,\ z \in E. \qquad (1.2)$$

It is noted that for each class $P[A,B,\alpha]$, one can find $\widetilde{A}$ and $\widetilde{B}$ such that $P[A,B,\alpha] \equiv P[\widetilde{A},\widetilde{B}]$. Moreover, for given class $P[\widetilde{A},\widetilde{B}]$ there are infinite many A,B,α such that $P[A,B,\alpha] \equiv P[\widetilde{A},\widetilde{B}]$. Therefore, in (1.2) we can write

$$p_1(z) \prec \frac{1+Cz}{1+Bz},\ C \in \mathbb{C},\ B \in [-1,0],\ 0 \le \alpha < 1,\ z \in E,$$

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where $C = (1-\alpha)A+\alpha B$. It can also be noted that

$$(1-\alpha)p_1+\alpha \in P[A,B,\alpha] \Leftrightarrow p_1 \in P[A,B]. \qquad (1.3)$$

Now we generalize this concept of Janowski functions and define the class $P_k[A,B,\alpha]$ as follows.

A function *p* is said to be in the class $P_k[A,B,\alpha]$, if and only if,

$$p(z)=\left(\frac{k}{4}+\frac{1}{2}\right)p_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)p_2(z), \qquad (1.4)$$

where $p_1, p_2 \in P[A,B,\alpha]$, $A\in\mathbb{C}$, $B\in[-1,0]$, $k\geq 2$ and $0\leq\alpha<1$. It is clear that

$$P_2[A,B,\alpha]\equiv P[A,B,\alpha]$$

and $P_k[1,-1,0] \equiv P_k$, the well-known class given and studied by Pinchuk [12].

The important fact about the class $P_k[A,B,\alpha]$ is that this class is convex set. That is, for $p_i \in P_k[A,B,\alpha]$ and $\alpha_i \geq 0$ with $\sum_{i=1}^{n}\alpha_i = 1$, we have

$$\sum_{i=1}^{n}\alpha_i p_i \in P_k[A,B,\alpha]. \qquad (1.5)$$

This can be easily seen from (1.3), (1.4) and with the fact that the set P[A,B] is convex [7]. By using all these concepts, we consider the following classes.

$$R_k[A,B,\alpha]=\left\{f\in A: \mathrm{Re}\frac{zf'(z)}{f(z)}\in P_k[A,B,\alpha],\ z\in E\right\},$$

$$V_k[A,B,\alpha]=\{f\in A: zf'(z)\in R_k[A,B,\alpha],\ z\in E\},$$

where $A\in\mathbb{C}$, $B\in[-1,0]$, $k\geq 2$ and $0\leq\alpha<1$. For $\alpha = 0$ and $-1\leq B<A\leq 1$, the classes $V_k[A,B,\alpha]$ and $R_k[A,B,\alpha]$ reduce to the classes $V_k[A,B]$ and $R_k[A,B]$ respectively, studied by Noor [4,5,8].

Throughout this article we assume that $C = (1-\alpha)A+\alpha B$ unless otherwise mentioned. In order to derive our main results, we need the following lemmas.

PRELIMINARY LEMMAS

Lemma 2.1: Let $\beta, \gamma, A\in\mathbb{C}$ with $\mathrm{Re}[\beta+\gamma]>0$ and $B\in[-1,0]$ satisfy either

$$\mathrm{Re}\left[\beta[1+CB]+\gamma(1+B^2)\right]\geq\left|C\beta+\bar{\beta}B+B(\gamma+\bar{\gamma})\right|, \qquad (2.1)$$

when $B\in(1,0]$, or

$$\mathrm{Re}\beta[1+C]>0 \text{ and } \mathrm{Re}\left[\beta(1-C)+2\gamma\right]\geq 0, \qquad (2.2)$$

when $B = -1$. If $h(z)=1+c_nz^n+c_{n+1}z^{n+1}+\ldots$ satisfies

$$\left\{h(z)+\frac{zh'(z)}{\beta h(z)+\gamma}\right\}\prec\frac{1+Cz}{1+Bz} \qquad (2.3)$$

then

$$h(z)\prec Q(z)\prec\frac{1+Cz}{1+Bz} \qquad (2.4)$$

and

$$Q(z)=\frac{1}{\beta G(z)}-\frac{\gamma}{\beta},$$

also

$$G(z)=\begin{cases}\dfrac{1}{n}\displaystyle\int_0^1\left[\frac{1+Btz}{1+Bz}\right]^{\frac{\beta(C-B)}{nB}}t^{\frac{\beta+\gamma}{n}-1}dt, & B\neq 0,\\[2ex] \dfrac{1}{n}\displaystyle\int_0^1 e^{\frac{\beta A}{n}(1-\alpha)(t-1)z}\,t^{\frac{\beta+\gamma}{n}-1}dt, & B=0,\end{cases}$$

or equivalently, in hypergeometric form, we can write it as

$$G(z)=\begin{cases}{}_2F_1\left(\dfrac{\beta(B-C)}{nB},1,\dfrac{\beta+\gamma+n}{n};\dfrac{Bz}{1+Bz}\right)(\beta+\gamma)^{-1}, & B\neq 0,\\[2ex] {}_1F_1\left(1,\dfrac{\beta+\gamma+n}{n};-\dfrac{\beta}{n}Az\right)(\beta+\gamma)^{-1}, & B=0.\end{cases}$$

The proof of this lemma can easily be obtained by using the similar argument used in [3, pp.109].

Lemma 2.2: [3] Let $u = u_1 + iu_2$, $v = v_1+iv_2$ and $\psi(u,v)$ be a complex valued function satisfying the conditions:

(i) $\psi(u,v)$ is continuous in a domain $D\subset\mathbb{C}^2$,
(ii) $(1,0)\in D$ and $\mathrm{Re}\psi(1,0)>0$,
(iii) $\mathrm{Re}\psi(iu_2, v_1)\leq 0$, whenever $(iu_2, v_1)\in D$ and $v_1\leq-\frac{1}{2}(1+u_2^2)$.

If $h(z) = 1+c_1z+\ldots$ is a function analytic in *E* such that $(h(z), zh'(z))\in D$ and $\mathrm{Re}\psi(h(z), zh'(z))>0$ for $z\in E$, then $\mathrm{Re}h(z)>0$ in *E*.

Lemma 2.3: [14]. Let ρ be a positive measure on [0,1] and let $h(z,t)$ be a complex-valued function defined on $E\times[0,1]$ such that $h(.,t)$ is analytic in E for each $t\in[0,1]$ and that $h(z,.)$ is ρ-integrable on [0,1] for all $z\in E$. In addition, suppose that Re $\{h(z,t)\}>0$, $h(-r,t)$ is real and

$$\mathrm{Re}\left\{\frac{1}{h(z,t)}\right\}\geq\frac{1}{h(-r,t)}\ \text{for}\ |z|\leq r<1\ \text{and}\ t\in[0,1].$$

If $H(z)=\int_0^1 h(z,t)d\rho(t)$, then $\mathrm{Re}\left(\frac{1}{H(z)}\right)\geq\frac{1}{H(-r)}$.

Lemma 2.4: [3]. For real or complex numbers a, b, c ($c\neq 0,-1,-2,\ldots$) and Rec>Reb>0, we have

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,b,c;z), \tag{2.5}$$

$${}_2F_1(a,b,c;z)=(1-z)^{-a}\,{}_2F_1\left(a,c-b,c;\frac{z}{z-1}\right), \tag{2.6}$$

$${}_2F_1(a,b,c;z)={}_2F_1(b,a,c;z). \tag{2.7}$$

MAIN RESULTS

Theorem 3.1: Let $f\in V_k[A,B,\alpha]$ with $k\geq 2$, $0\leq\alpha<1$ and $A\in\mathbb{C}, B\in[-1,0]$ satisfying (2.1) and (2.2). Then, $f\in R_k[A,B,\beta_1]$, where

$$\mathrm{Re}A<-\frac{B(1+\alpha)}{1-\alpha}, B\in[-1,0)$$

and

$$\beta_1=\beta_1(\alpha,1,0)=\frac{C}{(1-B)^{\frac{B-C}{B}}-(1-B)}. \tag{3.1}$$

Proof: Let

$$\frac{zf'(z)}{f(z)}=p(z)=\left(\frac{k}{4}+\frac{1}{2}\right)p_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)p_2(z). \tag{3.2}$$

Logarithmic differentiation of (3.2) yields

$$\frac{(zf'(z))'}{f'(z)}=p(z)+\frac{zp'(z)}{p(z)}.$$

Since $f\in V_k[A,B,\alpha]$, it follows that

$$p(z)+\frac{zp'(z)}{p(z)}\in P_k[A,B,\alpha]. \tag{3.3}$$

Now consider a function $\varphi_{a,b}(z)$ defined by Noor [6]

$$\varphi_{a,b}(z)=z+\sum_{n=2}^{\infty}\frac{b+1}{b+(n-1)a}z^n$$

with a = 1, b = 0 and then by using the same convolution technique as used by Noor [6], we have

$$\frac{\varphi_{a,b}(z)}{z}*p(z)=p(z)+\frac{zp'(z)}{p(z)}. \tag{3.4}$$

From (3.2), (3.3) and (3.4), we obtain

$$\begin{aligned}&\left(\frac{k}{4}+\frac{1}{2}\right)\left(p_1(z)+\frac{zp_1'(z)}{p_1(z)}\right)\\&-\left(\frac{k}{4}-\frac{1}{2}\right)\left(p_2(z)+\frac{zp_2'(z)}{p_2(z)}\right)\in P_k[A,B,\alpha].\end{aligned} \tag{3.5}$$

From this, we have

$$p_i(z)+\frac{zp_i'(z)}{p_i(z)}\in P[A,B,\alpha],\quad i=1,2.$$

We use Lemma 2.1 for n = 1, γ = 0, β = 1>0, $\alpha\in[0,1)$ and $h=p_i$ in (2.3) to have

$$p_i(z)\prec Q(z)\prec\frac{1+Cz}{1+Bz}. \tag{3.6}$$

This estimate is best possible, extremal function Q(z) is given by

$$Q(z)=\frac{1}{G(z)}=\begin{cases}\dfrac{Cz}{(1+Bz)-(1+Bz)^{\frac{B-C}{B}}}, & \text{if } B\neq 0,\\[2ex] \dfrac{(1-\alpha)Az}{1-e^{-(1-\alpha)Az}}, & \text{if } B=0.\end{cases} \tag{3.7}$$

From (3.6), we have

$$\underset{|z|=r}{\mathrm{Min}}\,\mathrm{Re}\,p_i(z)\geq\underset{|z|=r}{\mathrm{Min}}\,\mathrm{Re}\,Q(z).$$

Now we show that min Re Q(z) = Q(-1). Setting

$$a=\frac{B-C}{B}, b=1, c=b+1$$

such that Rec>Reb>0 and using (2.5), (2.6) and (2.7), we have

$$\begin{aligned}G(z)&=(1+Bz)^a\int_0^1 t^{b-1}(1+Btz)^{-a}dt\\&=\frac{\Gamma(b)}{\Gamma(c)}\,{}_2F_1\left(1,a,c;\frac{Bz}{1+Bz}\right),\ B\neq 0.\end{aligned} \tag{3.8}$$

Now we have to show that $\operatorname{Re}\{1/G(z)\} \geq 1/G(-1),\ z \in E$. For

$$\operatorname{Re}A < \frac{-B(1+\alpha)}{1-\alpha}$$

with $-1 \leq B < 0$ ($\operatorname{Re}c > \operatorname{Re}a > 0$) and using (2.5) in (3.8), we have

$$G(z) = \int_0^1 g(z,t)\,d\rho(t),$$

where

$$g(z,t) = \frac{1+Bz}{1+(1-t)Bz}$$

and

$$d\rho(t) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(c-a)} t^a (1-t)^{c-a-1}\,dt$$

is a positive measure on [0,1]. Now for $-1 \leq B < 0$, we have $\operatorname{Re} g(z,t) > 0$ and $g(-r,t)$ is real for $0 \leq r < 1 \leq$, $t \in [0,1]$. Also for $|z| \leq r < 1$ and $t \in [0,1]$

$$\operatorname{Re}\left\{\frac{1}{g(z,t)}\right\} = \operatorname{Re}\left\{\frac{1+(1-t)Bz}{1+Bz}\right\} \geq \frac{1-(1-t)Br}{1-Br} = \frac{1}{g(-r,t)}.$$

Now using the lemma 2.3, we obtain $\operatorname{Re}\{1/G(z)\} \geq 1/G(-r)$, ($|z| \leq r < 1$). Letting $r \to 1^-$, we obtain $\operatorname{Re}\{1/G(z)\} \geq 1/G(-1)$. Taking

$$\operatorname{Re}A \to \left(\frac{-B(1+\alpha)}{1-\alpha}\right)^+$$

and using (3.6), we have consequently from (3.7),

$$\beta_1 = \beta_1(\alpha,1,0) = Q(-1) = \frac{C}{(1-B)^{\frac{B-C}{B}} - (1-B)},$$

$$\operatorname{Re}A < -\frac{B(1+\alpha)}{1-\alpha},\ B \in [-1,0).$$

This shows that $p \in P[A,B,\beta_1]$ where β_1 is given by (3.1) and consequently $p \in P_k[A,B,\beta_1]$ which gives the required result.

If A = 1, B = -1 in Theorem 3.1, we obtain the following result, proved in [10].

Corollary 3.2: Let $f \in V_k(\alpha)$. Then $f \in R_k(\beta_1)$, where

$$\beta_1 = \beta_1(\alpha,1,0) = \begin{cases} \dfrac{2\alpha-1}{2-2^{2(1-\alpha)}}, & \text{if } \alpha \neq \dfrac{1}{2}, \\ \dfrac{1}{2\ln 2}, & \text{if } \alpha = \dfrac{1}{2}. \end{cases} \tag{3.9}$$

If k = 2, A = 1, B = -1 in Theorem 3.1, we obtain the following result, proved in [1].

Corollary 3.3: Let $f \in C(\alpha)$. Then $f \in S^*(\beta_1)$, where

$$\beta_1 = \beta_1(\alpha,1,0) = \begin{cases} \dfrac{2\alpha-1}{2-2^{2(1-\alpha)}}, & \text{if } \alpha \neq \dfrac{1}{2}, \\ \dfrac{1}{2\ln 2}, & \text{if } \alpha = \dfrac{1}{2}. \end{cases} \tag{3.10}$$

Theorem 3.4: Let $f \in V_k[A,B,\alpha]$. Then $f \in R_k[1,-1,\beta_2]$, where β_2 is one of the root of

$$\begin{aligned} &4(B^2-1)\beta_2^4 + \left[\begin{array}{l} 4(1-2\alpha)(B^2-1) \\ -8(1-\alpha)(AB-1) \end{array}\right]\beta_2^3 + \\ &\left[\begin{array}{l} (4\alpha^2-4\alpha-3)(B^2-1)+4(1-\alpha)^2(A^2-1) \\ -4(1-\alpha)(1-2\alpha)(AB-1) \end{array}\right]\beta_2^2 \\ &+\left[\begin{array}{l} 4(1-\alpha)(AB-1) \\ -2(1-2\alpha)(B^2-1) \end{array}\right]\beta_2 + (B^2-1) = 0 \end{aligned} \tag{3.11}$$

with $0 \leq \beta_2 < 1$.

Proof: Let

$$\frac{zf'(z)}{f(z)} = (1-\beta_2)p(z) + \beta_2 \tag{3.12}$$

$$= (1-\beta_2)\left[\left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z)\right] + \beta_2, \tag{3.13}$$

$p(z)$ is analytic in E with $p(0) = 1$. Then

$$\frac{(zf'(z))'}{f'(z)} = (1-\beta_2)p(z) + \beta_2 + \frac{(1-\beta_2)zp'(z)}{(1-\beta_2)p(z)+\beta_2},$$

that is,

$$\frac{1}{1-\alpha}\left[\frac{(zf'(z))'}{f'(z)} - \alpha\right] = \frac{1}{1-\alpha}\left[(1-\beta_2)p(z) + \beta_2 - \alpha + \frac{(1-\beta_2)zp'(z)}{(1-\beta_2)p(z)+\beta_2}\right]$$

$$= \frac{(\beta_2-\alpha)}{1-\alpha} + \frac{(1-\beta_2)}{1-\alpha}\left[p(z) + \frac{\frac{1}{(1-\beta_2)}zp'(z)}{p(z)+\frac{\beta_2}{(1-\beta_2)}}\right].$$

Since $f \in V_k[A,B,\alpha]$, it implies that

$$\frac{(\beta_2-\alpha)}{1-\alpha}+\frac{(1-\beta_2)}{1-\alpha}\left[p(z)+\frac{\frac{1}{(1-\beta_2)}zp'(z)}{p(z)+\frac{\beta_2}{(1-\beta_2)}}\right]\in P_k[A,B],\ z\in E. \tag{3.14}$$

Now consider a function $\varphi_{a,b}(z)$ defined by Noor [6]

$$\varphi_{a,b}(z)=z+\sum_{n=2}^{\infty}\frac{b+1}{b+(n-1)a}z^n \tag{3.15}$$

with

$$a=\frac{1}{1-\beta_2},\ b=\frac{\beta_2}{1-\beta_2}.$$

By using (3.7) with the same convolution technique as used by Noor [6], we have

$$\frac{\varphi_{a,b}(z)}{z}*p(z)=\left(\frac{k}{4}+\frac{1}{2}\right)\left[\frac{\varphi_{a,b}(z)}{z}*p_1(z)\right]-\left(\frac{k}{4}-\frac{1}{2}\right)\left[\frac{\varphi_{a,b}(z)}{z}*p_2(z)\right]$$

which implies that

$$p(z)+\frac{azp'(z)}{p(z)+b}=\left(\frac{k}{4}+\frac{1}{2}\right)\left[p_1(z)+\frac{azp_1'(z)}{p_1(z)+b}\right]-\left(\frac{k}{4}-\frac{1}{2}\right)\left[p_2(z)+\frac{azp_2'(z)}{p_2(z)+b}\right]. \tag{3.16}$$

Thus, from (3.14) and (3.16), we have

$$\frac{(\beta_2-\alpha)}{1-\alpha}+\frac{(1-\beta_2)}{1-\alpha}\left[p_i(z)+\frac{azp_i'(z)}{p_i(z)+b}\right]\in P[A,B],\ i=1,2. \tag{3.17}$$

Using the fact illustrated in (1.1), we have

$$\frac{(B-1)\left[(\lambda+\mu p_i(z))(p_i(z)+b)+a\mu zp_i'(z)\right]-(A-1)(p_i(z)+b)}{(B+1)\left[(\lambda+\mu p_i(z))(p_i(z)+b)+a\mu zp_i'(z)\right]-(A+1)(p_i(z)+b)}\in P,$$

where

$$\lambda=\frac{\beta_2-\alpha}{1-\alpha}\ \text{and}\ \mu=\frac{1-\beta_2}{1-\alpha}.$$

We now form the functional $\psi(u, v)$ by choosing $u = p_i(z)$, $v = zp_i'(z)$ and note that the first two conditions of Lemma 2.2 are clearly satisfied. We check condition (iii) as follows.

$$\psi(u,v)=\frac{(B-1)\left[(\lambda+\mu u)(u+b)+a\mu v\right]-(A-1)(u+b)}{(B+1)\left[(\lambda+\mu u)(u+b)+a\mu v\right]-(A+1)(u+b)}=\frac{\lambda_1+a\mu(B-1)v+\left[(\lambda+\mu(u+b))(B-1)-(A-1)\right]u}{\lambda_2+a\mu(B+1)v+\left[(\lambda+\mu(u+b))(B+1)-(A+1)\right]u},$$

where $\lambda_1=b[\lambda(B-1)-(A-1)]$ and $\lambda_2=b[\lambda(B+1)-(A+1)]$. Now

$$\psi(iu_2,v_1)=\frac{\lambda_1+\mu(av_1-u_2^2)(B-1)+\left[(\lambda+\mu b)(B-1)-(A-1)\right]iu_2}{\lambda_2+\mu(av_1-u_2^2)(B+1)+\left[(\lambda+\mu b)(B+1)-(A+1)\right]iu_2}.$$

Taking real part of $\psi(iu_2, v_1)$, we have

$$\operatorname{Re}\psi(iu_2, v_1) = \frac{-\left[-\lambda_1 + \mu\left(av_1 - u_2^2\right)(1-B)\right]\left[\lambda_2 + \mu\left(av_1 - u_2^2\right)(1+B)\right]}{\left[\lambda_2 + \mu\left(av_1 + u_2\right)(1+B)\right]^2 + \left[(\lambda+\mu b)(B+1) - (A+1)\right]^2 u_2^2}$$

$$+\frac{\left[(\lambda+\mu b)(B-1) - (A-1)\right]\left[(\lambda+\mu b)(B+1) - (A+1)\right]u_2^2}{\left[\lambda_2 + \mu\left(av_1 + u_2\right)(1+B)\right]^2 + \left[(\lambda+\mu b)(B+1) - (A+1)\right]^2 u_2^2}.$$

As a>0, μ>0, so applying $v_1 \le -\frac{1}{2}\left(1+u_2^2\right)$ and after a little simplification, we have

$$\operatorname{Re}\psi(iu_2, v_1) \le \frac{A_1 + B_1 u_2^2 + C_1 u_2^4}{D_1}, \tag{3.18}$$

where

$$A_1 = \frac{1}{4}\left[2\lambda_1 - a\mu(B-1)\right]\left[2\lambda_2 - a\mu(B+1)\right],$$

$$B_1 = -\frac{1}{2}\mu(a+2)\left[\lambda_1(B+1) - a\mu\left(B^2-1\right) + \lambda_2(B-1)\right] + (\lambda+\mu b)^2\left(B^2-1\right) - 2(\lambda+\mu b)(AB-1) + \left(A^2-1\right),$$

$$C_1 = -\frac{1}{4}\mu^2\left(1-B^2\right)(a+2)^2$$

and

$$D_1 = \left[\lambda_2 + \mu\left(av_1 + u_2\right)(1+B)\right]^2 + \left[(\lambda+\mu b)(B+1) - (A+1)\right]^2 u_2^2.$$

The right hand side of (3.18) is negative if $A_1 \le 0$ and $B_1 \le 0$. From $A_1 \le 0$, we have β_2 to be one of the root of

$$\begin{aligned}
&4\left(B^2-1\right)\beta_2^4 + \left[4(1-2\alpha)\left(B^2-1\right) - 8(1-\alpha)(AB-1)\right]\beta_2^3 + \\
&\left[\left(4\alpha^2 - 4\alpha - 3\right)\left(B^2-1\right) + 4(1-\alpha)^2\left(A^2-1\right) - 4(1-\alpha)(1-2\alpha)(AB-1)\right]\beta_2^2 \\
&+\left[4(1-\alpha)(AB-1) - 2(1-2\alpha)\left(B^2-1\right)\right]\beta_2 + \left(B^2-1\right) = 0
\end{aligned}$$

with $0 \le \beta_2 < 1$ and also for $0 \le \beta_2 < 1$, we have

$$\begin{aligned}
B_1 = {} & 4\left(B^2-1\right)\beta_2^3 + \left[4(1+\alpha)\left(B^2-1\right) + (1-\alpha)(AB-1)\right]\beta_2^2 - \\
&\left[\left(2\alpha^2 + 2\alpha + 5\right)\left(B^2-1\right) + 2(1+2\alpha)(1-\alpha)(AB-1) + 2\left(A^2-1\right)(1-\alpha)^2\right]\beta_2 + \\
&\left[\left(3+2\alpha^2\right)\left(B^2-1\right) + 4\alpha(1-\alpha)(AB-1) + 2\left(A^2-1\right)(1-\alpha)^2\right] \le 0.
\end{aligned}$$

Since all the conditions of Lemma 2.2 are satisfied, it follows that $p_i \in P$ in E for i = 1, 2 and consequently $p \in P_k[1,-1]$ and hence $f \in R_k[1, -1, \beta_2]$, where β_2 is one of the root of (3.11) with $0 \le \beta_2 < 1$.

By setting A = 1, B = -1 in Theorem 3.4, we obtain the following result, proved in [10].

Corollary 3.5: Let $f \in V_k(\alpha)$. Then $f \in R_k(\beta_2)$, where β_2 is one of the root of

$$2\beta_2^2 - (2\alpha-1)\beta_2 - 1 = 0 \text{ with } 0 \le \beta_2 < 1$$

which is

$$\beta_2 = \frac{1}{4}\left[(2\alpha-1) + \sqrt{4\alpha^2 - 4\alpha + 9}\right].$$

If k = 2, A = 1, B = -1 in Theorem 3.4, we obtain the following result, proved in [1].

Corollary 3.6: Let $f \in C(\alpha)$. Then $f \in S^*(\beta_2)$ where β_2 is one of the root of

$$2\beta_2^2 - (2\alpha-1)\beta_2 - 1 = 0 \text{ with } 0 \le \beta_2 < 1$$

which is

$$\beta_2 = \frac{1}{4}\left[(2\alpha-1) + \sqrt{4\alpha^2 - 4\alpha + 9}\right].$$

Application of Theorem 3.1

Theorem 3.7: Let f and g belong to $V_k[A,B,\alpha]$ with $k\geq 2$, $0\leq\alpha<1$ and $A\in\mathbb{C}, B\in[-1,0]$ satisfying (2.1) and (2.2). Then the function F(z) defined by

$$\left(F(z)\right)^{\alpha_1}=\frac{c}{z^{c-\alpha_1}}\int_0^z t^{(c-\delta-\upsilon)-1}\left(f(t)\right)^{\delta}\left(g(t)\right)^{\upsilon}dt, \alpha_1, c, \delta, \upsilon>0 \quad (3.19)$$

is in the class $R_k[A,B,\delta_1]$, where

$$\delta_1=\frac{c}{\alpha_1\ {}_2F_1\left(1,\alpha_1(1-\alpha)\left(1-\frac{A}{B}\right),c+1;\frac{B}{B-1}\right)}-\frac{c-\alpha_1}{\alpha_1} \quad (3.20)$$

and

$$-1\leq B<0, \operatorname{Re}A<-\frac{\left[(c+1)-\alpha_1(1-\alpha)\right]B}{\alpha_1(1-\alpha)}, \ (\delta+\upsilon)=\alpha_1.$$

Proof: Let

$$\frac{\left(zF'(z)\right)'}{F'(z)}=p(z) \quad (3.21)$$

$$=\left(\frac{k}{4}+\frac{1}{2}\right)p_3(z)-\left(\frac{k}{4}-\frac{1}{2}\right)p_4(z). \quad (3.22)$$

Then, p is analytic in E and p(0) = 1 with

$$\frac{zf'(z)}{f(z)}=p_1(z), \frac{zg'(z)}{g(z)}=p_2(z),$$

we have from (3.19) and by using (3.21)

$$p(z)+\frac{zp'(z)}{\alpha_1 p(z)+(c-\alpha_1)}=\frac{\delta}{\alpha_1}p_1(z)+\frac{\upsilon}{\alpha_1}p_2(z)=H_0(z).$$

Since f, $g\in V_k[A,B,\alpha]$ and this means that f, $g\in R_k[A,B,\beta_1]$, so p_1, $p_2\in P_k[A,B,\beta_1]$. It is known that $P_k[A,B,\beta_1]$ is convex set. Therefore, $H_0\in P_k[A,B,\beta_1]$ and $\beta_1=\beta_1(\alpha)$ is given by (3.1). This implies that

$$p(z)+\frac{zp'(z)}{\alpha_1 p(z)+(c-\alpha_1)}\in P_k[A,B,\beta_1]. \quad (3.23)$$

Now consider a function $\varphi_{a,b}(z)$ defined by Noor [6]

$$\varphi_{a,b}(z)=z+\sum_{n=2}^{\infty}\frac{b+1}{b+(n-1)a}z^n$$

with

$$a=\alpha_1,\ b=\frac{c-\alpha_1}{\alpha_1}$$

and using (3.22) with the same convolution technique as used by Noor [6], we have

$$\frac{\varphi_{a,b}(z)}{z}*p(z)=\left(\frac{k}{4}+\frac{1}{2}\right)\left[\frac{\varphi_{a,b}(z)}{z}*p_3(z)\right]-\left(\frac{k}{4}-\frac{1}{2}\right)\left[\frac{\varphi_{a,b}(z)}{z}*p_4(z)\right],$$

which implies that

$$\begin{aligned}p(z)&+\frac{zp'(z)}{\alpha_1 p(z)+(c-\alpha_1)}\\&=\left(\frac{k}{4}+\frac{1}{2}\right)\left[p_3(z)+\frac{zp_3'(z)}{\alpha_1 p_3(z)+(c-\alpha_1)}\right]\\&-\left(\frac{k}{4}-\frac{1}{2}\right)\left[p_4(z)+\frac{zp_4'(z)}{\alpha_1 p_4(z)+(c-\alpha_1)}\right].\end{aligned} \quad (3.24)$$

Thus, from (3.23) and (3.24), we have

$$p_i(z)+\frac{zp_i'(z)}{\alpha_1 p_i(z)+(c-\alpha_1)}\in P[A,B,\beta_1], \ i=3,4.$$

Therefore,

$$p_i(z)+\frac{zp_i'(z)}{\alpha_1 p_i(z)+(c-\alpha_1)}\prec\frac{1+\{(1-\beta_1)A+\beta_1 B\}z}{1+Bz}.$$

Using Lemma 2.1 for n = 1, $\beta=\alpha_1$ and $\gamma=c-\alpha_1$, we have

$$p_i(z)\prec Q(z)\prec\frac{1+\{(1-\beta_1)A+\beta_1 B\}z}{1+Bz},$$

where

$$Q(z)=\frac{1}{\alpha_1 G(z)}-\frac{c-\alpha_1}{\alpha_1}$$

and

$$G(z)=\begin{cases}{}_2F_1\left(1,\alpha_1\left(\frac{B-C}{B}\right),c+1;\frac{Bz}{1+Bz}\right)(c)^{-1}, & \text{if } B\neq 0,\\ {}_1F_1(1,c+1;-\alpha_1 Az)(c)^{-1}, & \text{if } B=0.\end{cases}$$

Now using the Lemma 2.3, we have $p\in P_k[A,B,\delta_1]$ where δ_1 is given by (3.20). This shows that $F\in R_k[A,B,\delta_1]$.

Theorem 3.8: Let f and g belong to $V_k[A,B,\alpha]$ with $k\geq 2$, $0\leq\alpha<1$ and $A\in\mathbb{C}$, $B\in[-1,0]$ satisfying (2.1) and

(2.2). Then, the function F with $\alpha_1 = c = 1$ defined by (3.19) is in the class $V_k[A,B,\eta]$, where $0\leq\delta<\upsilon\leq 1$,

$$\eta = \eta(\alpha) = \left(1-(\delta+\upsilon)(1+\beta_1)\right) \quad (3.25)$$

and $\beta_1(\alpha)$ is given by (3.1).

Proof: From (3.19), we can easily write

$$\frac{(zF'(z))'}{F'(z)} = \delta\frac{zf'(z)}{f(z)} + \upsilon\frac{zg'(z)}{g(z)} + 1-(\delta+\upsilon). \quad (3.26)$$

Since f and g belong to $V_k[A,B,\alpha]$ then by Theorem 3.1, $\frac{zf'(z)}{f(z)}$ and $\frac{zg'(z)}{g(z)}$ belong to $P_k[A,B,\beta_1]$ where $\beta_1 = \beta_1(\alpha)$ is given by (3.1). Using

$$\frac{zf'(z)}{f(z)} = (1-\beta_1)p_1(z)+\beta_1,\ p_1\in P_k[A,B]$$

and

$$\frac{zg'(z)}{g(z)} = (1-\beta_1)p_2(z)+\beta_1,\ p_2\in P_k[A,B]$$

in (3.26), we have

$$\frac{1}{1-\eta}\left[\frac{(zF'(z))'}{F'(z)}-\eta\right] = \frac{\delta}{\delta+\upsilon}p_1(z)+\frac{v}{\delta+\upsilon}p_2(z). \quad (3.27)$$

Now by using the well known fact that the class $P_k[A,B]$ is a convex set together with (3.27), we obtain the required result.

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