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On Some Topological Properties of Generalized Difference Sequence Spaces Defined by A Sequence of Moduli

[1]*Gülcan Atici and* [2]*Çigdem A. Bektas*

[1]Department of Mathematics, Mus Alparslan University, Mus, 49100, Turkey
[2]Department of Mathematics, Firat University, Elazig, 23119, Turkey

Abstract: In this paper, we define the generalized difference sequence spaces $\ell_\infty(\Delta_v^m,F,p,q,u)$, $c(\Delta_v^m,F,p,q,u)$ and $c_0(\Delta_v^m,F,p,q,u)$. We give various properties and some inclusions on this spaces. Furthermore we study some of their properties, such as solidity, symmetricity, convergence free etc.

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INTRODUCTION

Let w be the set of all sequences of real or complex numbers and l_∞, c and c_0 be the linear spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|$$

where $k \in IN = \{1,2,...\}$, the set of positive integers.
The difference sequence spaces

$$X(\Delta) = \{x = (x_k) : \Delta x \in X\}$$

first defined by Kizmaz [5], where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$ and X is any of the sets $X = c_0$, c and l_∞. The notion of difference sequence spaces was generalized by Et and Çolak [6] as follows:

$$X(\Delta^m) = \{x = (x_k) : \Delta^m x \in X\}$$

for $X = c_0$, c and l_∞, where $m \in IN$, $\Delta^0 x = (x_k)$, $\Delta^m x_k = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$.

The sequence spaces $X(\Delta^m)$ were further generalized by Et and Esi [7] to following sequence spaces. Let $\nu = (\nu_k)$ be any fixed sequence of nonzero complex numbers. Then

$$X(\Delta_v^m) = \{x = (x_k) : (\Delta_v^m x_k) \in X\}$$

for $X = l_\infty$, c and c_0, where

$$\Delta_v^0 x = (v_k x_k)$$

$$\Delta_v x_k = (v_k x_k - v_{k+1} x_{k+1})$$

and

$$\Delta_v^m x_k = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})$$

and so that

$$\Delta_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}$$

A function $f:[0,\infty) \to [0,\infty)$ is called a modulus function if

(i) $f(t) = 0$ iff $t = 0$,
(ii) $f(t+u) \le f(t) + f(u)$, $\forall t, u \ge 0$
(iii) f is increasing,
(iv) f is continuous from the right at 0.

Since $|f(x)-f(y)| \le f(|x-y|)$, it follows from condition (iv) that f is continuous on $[0,\infty)$. A modulus may be unbounded or bounded.

Ruckle [8] used the idea of a modulus function to construct some spaces of complex sequences. Maddox [9] and Et [4] investigated and discussed some properties of some sequence spaces defined using a modulus function f. Recently Bektas and Çolak [2, 3] used a sequence of moduli $F = (f_k)$ to define some sequence spaces and introduced some new sequence spaces by using a sequence of moduli $F = (f_k)$

Definition 1.1: [10] Let X be a sequence space. Then X is called:

Corresponding Author: Dr. Gülcan Atici, Department of Mathematics, Mus Alparslan University, Mus, 49100, Turkey

(i) Solid (or normal), if $(\alpha_k x_k)\in X$ whenever $(x_k)\in X$ for all sequences (α_k) of scalars with $|\alpha_k|\le 1$.
(ii) Symmetric if $(x_k)\in X$ implies $(x_{\pi(k)})\in X$, where $\pi(k)$ is a permutation of $\mathbb{N}$.
(iii) A sequence algebra if (x_k), $(y_k)\in X$ implies $(x_k y_x)\in X$.
(iv) Convergence free if $(y_k)\in X$ whenever $(x_k)\in X$ and $y_k=\theta$ whenever $x_k=\theta$.

Let U be the set of all real sequences $u = (u_k)$ such that $u_k>0$ for all $k\in\mathbb{N}$.
We use the following inequality throughout this paper

$$|a_k + b_k|^{p_k} \le D\{|a_k|^{p_k} + |b_k|^{p_k}\} \qquad (1)$$

where a_k and b_k are complex numbers, $D = \max(1, 2^{G-1})$ and $H = \sup_k p_k < \infty$ [11].

MAIN RESULTS

Definition 2.1: Let $F = (f_k)$ be a sequence of moduli, X be a seminormed space over the field C of complex numbers with the seminorm q, $p = (p_k)$ be a sequence of strictly positive real numbers and $u\in U$. By w(X) we shall denote the space of all sequences defined over X. Let $\nu = (\nu_k)$ be any fixed sequence of nonzero complex numbers. Now we define the following sequence spaces:

$$\ell_\infty(\Delta_v^m,F,p,q,u)=\left\{x\in w(X): \sup_k u_k[f_k(q(\Delta_v^m x_k))]^{p_k}<\infty\right\}$$

$$c(\Delta_v^m,F,p,q,u)=\left\{\begin{array}{l} x\in w(X): \lim_{k\to\infty} u_k[f_k(q(\Delta_v^m x_k-\ell))]^{p_k}=0, \\ \text{for some } \ell \end{array}\right\}$$

$$c_0(\Delta_v^m,F,p,q,u)=\left\{x\in w(X): \lim_{k\to\infty} u_k[f_k(q(\Delta_v^m x_k))]^{p_k}=0\right\}$$

For $p_k = 1$ for all $k\in\mathbb{N}$, we write these spaces as $\ell_\infty(\Delta_v^m,F,p,q)$, $c(\Delta_v^m,F,q,u)$ and $c_0(\Delta_v^m,F,q,u)$.

For $u_k = 1$ for all $k\in\mathbb{N}$, we write these spaces as $\ell_\infty(\Delta_v^m,F,p,q)$, $c(\Delta_v^m,F,p,q)$ and $c_0(\Delta_v^m,F,p,q)$.

Theorem 2.1: Let $F = (f_k)$ be a sequence of moduli. Then $c_0(\Delta_v^m,F,p,q,u)\subset c(\Delta_v^m,F,p,q,u)\subset \ell_\infty(\Delta_v^m,F,p,q,u)$ and the inclusions are strict.

Proof: The first inclusion is obvious. We establish the second inclusion. Let $x\in c(\Delta_v^m,F,p,q,u)$. By definition of modulus function and inequality (1), we have

$$u_k[f_k(q(\Delta_v^m x_k))]^{p_k} \le Du_k[f_k(q(\Delta_v^m x_k-\ell))]^{p_k} + Du_k[f_k(q(\ell))]^{p_k}$$

Now we may choose an integer K_ℓ such that $q(\ell)\le K_\ell$. Hence, we have

$$u_k[f_k(q(\Delta_v^m x_k))]^{p_k} \le Du_k[f_k(q(\Delta_v^m x_k-\ell))]^{p_k} + \max[1,((K_\ell)f_k(1))^H]$$

Therefore, $x\in \ell_\infty(\Delta_v^m,F,p,q,u)$.

To show the inclusions are strict consider the following example.

Example 1: Let $f_k(x) = x$, $p_k = 1$, $\nu_k = 1$, $u_k = 1$ for all $k\in\mathbb{N}$ and $q(x) = |x|$. Then, the sequence $x = (k^m)$ belongs to $c(\Delta_v^m,F,p,q,u)$ but does not belong to $c_0(\Delta_v^m,F,p,q,u)$ and the sequence $x = ((-1)^k)$ belongs to $\ell_\infty(\Delta_v^m,F,p,q,u)$ but does not belong to $c(\Delta_v^m,F,p,q,u)$. Therefore the inclusions are strict.

Theorem 2.2: Let the sequence (p_k) be bounded. Then $\ell_\infty(\Delta_v^m,F,p,q,u)$, $c(\Delta_v^m,F,p,q,u)$ and $c_0(\Delta_v^m,F,p,q,u)$ are linear spaces over the complex field C.
The proof is easy and thus omitted.

Theorem 2.3: The space $c_0(\Delta_v^m,F,p,q,u)$ is a paranormed space, paranormed by

$$g(x)=\sum_{i=1}^m f_k(q(v_i x_i)) + \sup_k u_k[f_k(q(\Delta_v^m x_k))]^{p_k/M}$$

where $M = \max(1, \sup_k p_k)$; $c(\Delta_v^m,F,p,q,u)$ and $\ell_\infty(\Delta_v^m,F,p,q,u)$ are paranormed by g if $\inf p_k>0$.

The proof is routine verification by using standard arguments and therefore omitted.

Theorem 2.4: Let $F = (f_k)$ and $G = (g_k)$ be two sequences of moduli. For any two sequences $p = (p_k)$ and $t = (t_k)$ of strictly positive real numbers and seminorms q, q_1 and q_2 we have

(i) $Z(\Delta_v^m,G,p,q,u)\subset Z(\Delta_v^m,F\circ G,p,q,u)$
(ii) $Z(\Delta_v^m,F,p,q,u)\cap Z(\Delta_v^m,G,p,q,u)\subseteq Z(\Delta_v^m,F+G,p,q,u)$
(iii) $Z(\Delta_v^m,F,p,q_1,u)\cap Z(\Delta_v^m,F,p,q_2,u)\subset Z(\Delta_v^m,F,p,q_1+q_2,u)$
(iv) If q_1 is stronger than q_2 then $Z(\Delta_v^m,F,p,q_1,u)\subset Z(\Delta_v^m,F,p,q_2,u)$
(v) If q_1 is equivalent to q_2 then $Z(\Delta_v^m,F,p,q_1,u)=Z(\Delta_v^m,F,p,q_2,u)$
(vi) $Z(\Delta_v^m,F,p,q_1,u)\cap Z(\Delta_v^m,F,t,q_2,u)\ne\emptyset$

where $Z = l_\infty$, c or c_0.

Proof: (i) We prove for Z = c_0 and the rest cases follows similarly. Let $x \in c_0(\Delta_v^m,G,p,q,u)$ so that

$$\mu_k = u_k[g_k(q(\Delta_v^m x_k))]^{p_k} \to 0,(k \to \infty)$$

Let ε>0 and choose δ with 0<δ<1 such that $f_k(t)<\varepsilon$ for 0≤t≤δ. We write

$$s_1 = \{k \in IN : g_k(q(\Delta_v^m x_k)) \le \delta\}$$

$$s_2 = \{k \in IN : g_k(q(\Delta_v^m x_k)) > \delta\}$$

If $x \in c_0(\Delta_v^m,G,p,q,u)$, then for $g_k(q(\Delta_v^m x_k)) > \delta$

$$g_k(q(\Delta_v^m x_k)) < g_k(q(\Delta_v^m x_k))\delta^{-1} < 1+[g_k(q(\Delta_v^m x_k))\delta^{-1}]$$

where $k\in s_2$ and [n] denotes the integer part of n. Given ε>0, by definition of modulus function, we have, for $g_k(q(\Delta_v^m x_k)) > \delta$,

$$f_k(g_k(q(\Delta_v^m x_k))) \le \left(1+[g_k(q(\Delta_v^m x_k))]\delta^{-1}\right)f_k(1)$$
$$\le 2f_k(1)(g_k(q(\Delta_v^m x_k)))\delta^{-1}$$

and hence,

$$u_k[f_k(g_k(q(\Delta_v^m x_k)))]^{p_k} \le [2f_k(1)\delta^{-1}]^H \mu_k$$
$$< \varepsilon, (k \in s_2, k > k_2), \quad (2)$$

If $x \in c_0(\Delta_v^m,G,p,q,u)$, for $g_k(q(\Delta_v^m x_k)) \le \delta$,

$$f_k(g_k(q(\Delta_v^m x_k))) < \varepsilon$$

where $k\in s_1$. Given ε>0 if $k\in s_2$, we have

$$u_k[f_k(g_k(q(\Delta_v^m x_k)))]^{p_k} \le u_k \max(\varepsilon^{\inf_k p_k}, \varepsilon^{\sup_k p_k})$$
$$< \varepsilon, (k \in s_1, k > k_1) \quad (3)$$

from (2) and (3) for every $k>\max\{k_1,k_2\}$, $u_k[f_k(g_k(q(\Delta_v^m x_k)))]^{p_k} < \varepsilon$. Hence, $x \in c_0(\Delta_v^m, F\circ G,p,q,u)$. Thus, $c_0(\Delta_v^m,G,p,q,u) \subset c_0(\Delta_v^m, F\circ G,p,q,u)$.

(ii) It follows from the following inequality

$$u_k[(f_k+g_k)(q(\Delta_v^m x_k))]^{p_k} \le Du_k[f_k(q(\Delta_v^m x_k))]^{p_k}$$
$$+ Du_k[g_k(q(\Delta_v^m x_k))]^{p_k}$$

(iii) It follows from the following inequality

$$u_k[f_k(q_1+q_2)(\Delta_v^m x_k)]^{p_k} \le Du_k[f_k(q(\Delta_v^m x_k))]^{p_k}$$
$$+ Du_k[g_k(q_2(\Delta_v^m x_k))]^{p_k}$$

(iv), (v) and (vi) follow obviously.

Theorem 2.5: Let (X, q) be a complete seminormed space. Then, the spaces $\ell_\infty(\Delta_v^m,F,p,q)$, $c(\Delta_v^m,F,p,q)$ and $c_0(\Delta_v^m,F,p,q)$ are complete with the paranorm

$$g(x) = \sum_{i=1}^{m} f_k(q(v_i x_i)) + \sup_k [f_k(q(\Delta_v^m x_k))]^{p_k/M}$$

Proof: We give the proof for $\ell_\infty(\Delta_v^m,F,p,q)$ only. The other cases can be proved in a similar way. Let (x^s) be a Cauchy sequence in $\ell_\infty(\Delta_v^m,F,p,q)$ Then,

$$g(x^s - x^t) \to 0 \text{ as } s,t \to \infty$$

Then for each ε>0, there exists $n_0\in IN$ such that g $(x^s-x^t)<\varepsilon$, for all s,t>n_0.

Hence, $f_k(q(v_i x_i^s - v_i x_i^t)) < \varepsilon$ (i≤m) and $[f_k(q(\Delta_v^m(x_k^s - x_k^t)))]^{p_k/M} < \varepsilon$, for all s,t>$n_0$ and for each k Since each f_k is a modulus then $q(x_i^s - x_i^t) < \varepsilon$ $(i \le m)$ and $(q(\Delta_v^m(x_k^s - x_k^t))) < \varepsilon$, for all s,t>$n_0$.

Hence (x_i^s) (i≤m) and $(\Delta_v^m(x_k^s))$, for all k∈IN, are Cauchy sequences in X. Since X is complete, they are convergent in X. Suppose that $x_i^s \to x_i$ (i≤m) and $\Delta_v^m(x_k^s) \to y_k$, for all k∈IN, as s→∞.

Then, we can find a sequence (x_k) such that $y_k = \Delta_v^m x_k$ for all k∈IN. These x_k's can be written as

$$x_k = v_k^{-1}\sum_{j=1}^{k-m}(-1)^m \binom{k-j-1}{m-1} y_j$$
$$= v_k^{-1}\sum_{j=1}^{k}(-1)^m \binom{k+m-j-1}{m-1} y_{j-m}$$

where $y_{1-m} = y_{2-m} = ... = y_0 = 0$ for sufficiently large k, for instance, k>2m. Now using the continuity of f_k, we have for all s>n_0,

$$\sum_{i=1}^{m} f_k(q(v_i x_i^s - \lim_{t\to\infty} v_i x_i^t)) + \sup_k [f_k(q(\Delta_v^m(x_k^s - \lim_{t\to\infty} x_k^t))]^{p_k/M} < 2\varepsilon$$

Hence, $g(x^s-x)\to 0$ as r→∞. Since (x^s-x) and $(x^s) \in \ell_\infty(\Delta_v^m,F,p,q)$ and the sequence space $\ell_\infty(\Delta_v^m,F,p,q)$ is linear, we have

$$x = x^s - (x^s - x) \in \ell_\infty(\Delta_v^m,F,p,q).$$

Therefore $\ell_\infty(\Delta_v^m,F,p,q)$ is complete.

Theorem 2.6: Let $m\geq 1$ then for all $0<i\leq m$, $Z(\Delta_v^i,F,q,u)\subset Z(\Delta_v^m,F,q,u)$ where $Z = l_\infty$, c or c_0. The inclusions are strict.

Proof: We show that $\ell_\infty(\Delta_v^{i-1},F,q,u)\subset \ell_\infty(\Delta_v^i,F,q,u)$ for any $0<i\leq m$. It follows from the following inequality

$$u_k[f_k(q(\Delta_v^i x_k))]\leq u_k[f_k(q(\Delta_v^{i-1}x_k))]+u_k[f_k(q(\Delta_v^{i-1}x_{k+1}))]$$

that $(x_k)\in \ell_\infty(\Delta_v^{i-1},F,q,u)$ implies $(x_k)\in \ell_\infty(\Delta_v^i,F,q,u)$.

On applying the principle of induction, it follows that

$$\ell_\infty(\Delta_v^{i-1},F,q,u)\subset \ell_\infty(\Delta_v^m,F,q,u), \text{for } i=0,1,2,\ldots,m-1.$$

The proof for the rest cases are similar.

To show that the inclusions are strict consider the following example.

Example 2: Let X = C, $f_k(x) = x$, $p_k = 1$, $\nu_k = 1$, $u_k = 1$ for all $k\in$IN and $q(x) = |x|$. Then, the sequence $x=(k^m)\in Z(\Delta_v^m,F,q,u)$ but $x\notin Z(\Delta_v^{m-1},F,q,u)$ for $Z = l_\infty$ and c. Under the above restrictions, consider the sequence $(x_k) = (k^{m-1})$. Then $(x_k)\in Z(\Delta_v^m,F,q,u)$, but $(x_k)\notin Z(\Delta_v^{m-1},F,q,u)$ for $Z = c_0$. Therefore the inclusions are strict.

Theorem 2.7: Let $0<p_k\leq t_k$ and (t_k/p_k) be bounded. Then $Z(\Delta_v^m,F,t,q)\subset Z(\Delta_v^m,F,p,q)$ where $Z = l_\infty$, c or c_0.

Proof: We shall prove only $c_0(\Delta_v^m,F,t,q)\subset c_0(\Delta_v^m,F,p,q)$. The other inclusions can be proved similary. Let $x\in c_0(\Delta_v^m,F,t,q)$. Write $w_k=[f_k(q(\Delta_v^m x_k))]^{t_k}$ and $\lambda_k=\frac{p_k}{t_k}$, so that $0<\lambda<\lambda_k\leq 1$ for each k.

We define the sequences (r_k) and (s_k) as follows:

Let $r_k = w_k$ and $s_k = 0$ if $w_k\geq 1$ and let $r_k = 0$ and $s_k = w_k$ if $w_k<1$. Then it is clear that for all $k\in$IN, we have $w_k = r_k+s_k$, $w_k^{\lambda_k}=r_k^{\lambda_k}+s_k^{\lambda_k}$. Now it follows that $r_k^{\lambda_k}\leq r_k\leq w_k$ and $s_k^{\lambda_k}\leq s_k^{\lambda}$. Now we have

$$\lim_k w_k^{\lambda_k}\leq \lim_k w_k+(\lim_k s_k)^{\lambda}$$

This implies that $x\in c_0(\Delta_v^m,F,p,q)$ and this completes the proof.

Theorem 2.8: For any two sequences $p = (p_k)$ and $t = (t_k)$, we have $c_0(\Delta_v^m,F,t,q)\subset c_0(\Delta_v^m,F,p,q)$ if and only if $\liminf\frac{p_k}{t_k}>0$.

Proof: If we take $y_k=f_k(q(\Delta_v^m x_k))$ for all $k\in$IN, then by using the same technique of Lemma 1 of Maddox [12], it is easy to prove the theorem.

The following result is a consequence of the above theorem.

Theorem 2.9: For any two sequences $p = (p_k)$ and $t = (t_k)$, we have $c_0(\Delta_v^m,F,t,q)= c_0(\Delta_v^m,F,p,q)$ if and only if

$$\liminf\frac{p_k}{t_k}>0 \text{ and } \liminf\frac{t_k}{p_k}>0.$$

Theorem 2.10: The sequence spaces $c(\Delta_v^m,F,q,u)$ and $c_0(\Delta_v^m,F,q,u)$ are nowhere dense subsets of $\ell_\infty(\Delta_v^m,F,q,u)$.

Proof: It follows from Theorem 2.1.

Theorem 2.11: The sequence spaces $Z(\Delta_v^m,F,p,q,u)$ for $Z = l_\infty$, c or c_0 are not solid for m>0.

Proof: Let X = C, $f_k(x) = x$, $p_k = 1$, $u_k = 1$, $\nu_k = 1$, for all $k\in$IN and $q(x) = |x|$. Then $(x_k)=(k^m)\in \ell_\infty(\Delta_v^m,F,p,q,u)$ but $(\alpha_k x_k)\notin \ell_\infty(\Delta_v^m,F,p,q,u)$ when $\alpha_k = (-1)^k$ for all $k\in$IN. Hence $\ell_\infty(\Delta_v^m,F,p,q,u)$ is not solid. The other cases can be proved on considering similar examples.

Theorem 2.12: The sequence spaces $Z(\Delta_v^m,F,p,q,u)$ for $Z = l_\infty$, c or c_0 are not symmetric for m>0.

Proof: Under the restrictions on X, p, f_k, q, u and ν as given in the proof of Theorem 2.11, consider the sequence $(x_k) = (k^m)$, then $x\in Z(\Delta_v^m,F,p,q,u)$ for $Z = l_\infty$, c or c_0. Let (y_k) be a rearrangement of (x_k), which is defined as follows:

$$(y_k)=\{x_1,x_2,x_4,x_3,x_9,x_5,x_{16},x_6,x_{25},x_7,x_{36},x_8,x_{49},x_{10},\ldots\}$$

Then $(y_k)\notin Z(\Delta_v^m,F,p,q,u)$ for $Z = l_\infty$, c or c_0.

Theorem 2.13: The sequence space $c_0(\Delta_v^m,F,p,q,u)$ is not sequence algebra.

Proof: Under the restrictions on X, p, f_k, q, u and ν as given in the proof of Theorem 2.11, consider the sequence x = (k^{m-2}) and y = (k^{m-2}), then $x,y \in c_0(\Delta_v^m,F,p,q,u)$ but $x.y \notin c_0(\Delta_v^m,F,p,q,u)$.

Theorem 2.14: The sequence spaces $Z(\Delta_v^m,F,p,q,u)$ for $Z = l_\infty$, c or c_0 are not convergence free.

Proof: Let X = C, $f_k(x) = x$, for all $x\in[0,\infty)$, m = 1, $p_k = 1$, $u_k = 1$, $\nu_k = 1$, for all $k\in\mathbb{N}$ and $q(x) = |x|$. Consider the sequence $x_k = 1$, for all $k\in\mathbb{N}$. Then $(x_k) \in Z(\Delta)$ for $Z = l_\infty$, c or c_0. Consider the sequence (y_k) defined as $y_k = k^2$ for all $k\in\mathbb{N}$. Then the sequence (y_k) neither belongs to $c_0(\Delta)$nor to $c(\Delta)$ nor to $l_\infty(\Delta)$. Hence the sequence spaces are not convergence free.

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