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Direct Solution of Singular Higher-order BVPs by the Homotopy Analysis Method and its Modification

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Abstract: In this paper, series solutions of higher-order singular Boundary-value Problems (BVPs) are obtained directly by the Homotopy Analysis Method (HAM) and its modification (MHAM). The HAM and MHAM provide a convenient way of controlling the convergence region and rate of the series solution.

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Key words: Homotopy analysis method . modified homotopy analysis method . singular BVPs (series solution

INTRODUCTION

In this work, we consider a class of singular higher-order BVPs of the type

$$u^{(n+1)}(t)+\frac{r}{t}u^{(n)}(t)+f(t)u^{p}(t)=g(t), \quad n=1,2,3,\ldots,p=0,1,2,\ldots \tag{1}$$

subject to the boundary conditions

$$u(0)=a_0,\quad u'(0)=a_1,\ldots,u^{(n-1)}(0)=a_{n-1},\quad u(b)=c \tag{2}$$

where u^p is the nonlinear term, $r<0$, $f(t)$, $g(t)$ are continuous functions and the parameters $a_0, a_1, \ldots, a_{n-1}$, b, c are real constants.

Problems of the form (1)--(2) are encountered in the fields of fluid dynamics, elasticity, reaction-diffusion processes, chemical kinetics and other branches of applied mathematics. Singular two-point BVPs for ordinary differential equations arise very frequently in the theory of thermal explosions and in the study of electro-hydrodynamics [1]. Exact solutions of these problems are of great importance due to its wide applications in scientific research. Singular BVPs have been studied by several authors. Hasan and Zhu [2, 3] used the modified Adomian decomposition method (MADM) to find exact solutions of a certain class of singular boundary value problems. Various computational techniques for solving singular BVPs were given by Kumar and Singh [4]. Abu-Zaid and El-Gebeily [5] provided a finite difference approximation to the solution of a class of singular two-point boundary value problems. Kanth and Reddy [6] presented a method based on cubic splines for solving a class of singular two-point boundary value problems. The existence of a unique solution of (1)-(2) has been discussed in [5].

The homotopy analysis method (HAM), proposed by Liao in his Ph.D. thesis [7], is a powerful method to solve non-linear problems. In recent years, this method has been successfully employed to solve many types of nonlinear problems in science and engineering [8-22]. All of these successful applications verified the validity, effectiveness and flexibility of the HAM. A modification of the HAM was presented in [23]. Very recently, Bataineh *et al.* [24] applied the MHAM of [23] to get exact solutions of singular second-order two-point BVPs.

The aim of this paper is to apply the HAM and MHAM for the first time to obtain exact/approximate solutions of the singular higher-order BVPs of type (1)-(2).

HAM AND MHAM SOLUTIONS

Standard HAM: To solve (1)--(2) by means of HAM [7], we choose the general initial boundary approximation as

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$$u_0(t) \tag{3}$$

and the linear operator

$$\mathcal{L}[\phi(t;q)]=\frac{\partial^{n+1}\phi(t;q)}{\partial t^{n+1}}+\frac{r}{t}\frac{\partial^n\phi(t;q)}{\partial t^n} \tag{4}$$

with the property

$$\mathcal{L}\left[c_0+c_1t+\ldots+c_{n-1}t^{n-1}+c_nt^{n-r}\right]=0 \tag{5}$$

where c_i ($i = 0,1,\ldots n$) are constants of integration. Based on Eq. (1) we define a nonlinear operator

$$\mathcal{N}[\phi(t;q)]=\frac{\partial^{n+1}\phi(t;q)}{\partial t^{n+1}}+\frac{r}{t}\frac{\partial^n\phi(t;q)}{\partial t^n}+f(t)\phi^p(t;q) \tag{6}$$

Using the above definition, we construct the zeroth-order deformation equation as

$$(1-q)\mathcal{L}[\phi(t;q)-u_0(t)]=q\hbar\{\mathcal{N}[\phi(t;q)]-g(t)\} \tag{7}$$

where a nonzero auxiliary parameter is denoted by $\hbar$. Here $q\in[0,1]$ is an embedding parameter. Note that when q increases from 0 to 1, then $\phi(t;q)$ approaches from $u_0(t)$ to $u(t)$. Expanding $\phi(t;q)$ in Taylor series with respect to q, one has

$$\phi(t;q)=\sum_{m=0}^{+\infty}u_m(t)q^m \tag{8}$$

Now differentiating (7) m times with respect to the embedding parameter q and then setting q = 0 and finally dividing them by m!, we have the so-called *mth-order deformation equation*

$$\mathcal{L}[u_m(t)-\chi_m u_{m-1}(t)]=\hbar R_m(\vec{u}_{m-1}) \tag{9}$$

where [10, 11]

$$\begin{aligned}R_m(\vec{u}_{m-1})&=u_{m-1}^{(n+1)}+\frac{r}{t}u_{m-1}^{(n)}+f(t)\left[\sum_{r_1=0}^{m-1}u_{m-r_1-1}\sum_{r_2=0}^{r_1}u_{r_1-r_2}\sum_{r_3=0}^{r_2}u_{r_2-r_3}\cdots\right.\\&\left.\times\sum_{r_{p-2}=0}^{r_{p-3}}u_{r_{p-3}-r_{p-2}}\sum_{r_{p-1}=0}^{r_{p-2}}u_{r_{p-2}-r_{p-1}}u_{r_{p-1}}\right]-(1-\chi_m)g(t)\end{aligned} \tag{10}$$

and

$$\chi_m=\begin{cases}0, & m\le 1\\ 1, & m>1\end{cases}$$

and

$$\vec{u}_m(t)=\{u_0(t),u_1(t),\ldots,u_{m-1}(t)\}$$

It should be emphasized that $u_m(t)$ ($m\geq1$) is governed by the linear equations (9) with the linear boundary conditions that come from the original problem, which can be easily solved by symbolic computation softwares such as Maple and Mathematica.

Assuming that $\hbar$ is properly chosen so that the homotopy-series (8) is convergent at q = 1, we have the homotopy-series solution

$$u(t) = u_0(t)+\sum_{m=1}^{+\infty}u_m(t) \tag{11}$$

Modified HAM: The MHAM [23] suggests that the function g(t) can be divided into n terms, namely $k_0(t)+\ldots+k_n(t)$. According to MHAM we expand $\varphi(t;q)$ in powers of the embedding parameter q namely

$$\varphi(t;q)=k_0(t)q^0+k_1(t)q^1+\cdots+k_n(t)q^n \tag{12}$$

According to (7) and (12) the new zeroth-order deformation equation

$$(1-q)\mathcal{L}[\phi(t;q)-u_0(t)]=q\hbar\{\mathcal{N}[\phi(t;q)]-\varphi(t;q)\} \tag{13}$$

and the mth-order deformation equation is

$$\mathcal{L}[u_m(t)-\chi_m u_{m-1}(t)]=\hbar\mathcal{R}_m(\vec{u}_{m-1}) \tag{14}$$

where

$$\begin{aligned}\mathcal{R}_m(\vec{u}_{m-1})&=u_{m-1}^{(n+1)}+\frac{r}{t}u_{m-1}^{(n)}+f(t)\left[\sum_{r_1=0}^{m-1}u_{m-r_1-1}\sum_{r_2=0}^{r_1}u_{r_1-r_2}\sum_{r_3=0}^{r_2}u_{r_2-r_3}\cdots\right.\\&\left.\times\sum_{r_{p-2}=0}^{r_{p-3}}u_{r_{p-3}-r_{p-2}}\sum_{r_{p-1}=0}^{r_{p-2}}u_{r_{p-2}-r_{p-1}}u_{r_{p-1}}\right]-k_{m-1}(t)\end{aligned} \tag{15}$$

NUMERICAL EXPERIMENTS

To illustrate the effectiveness of the HAM and MHAM we shall consider three examples of linear and nonlinear singular BVPs.

Example 1: We first consider the singular two-point BVPs [3],

$$u''(t)+\frac{r}{t}u'(t)=g(t),\quad r<0 \tag{16}$$

subject to the boundary conditions

$$u(0)=0,\quad u(1)=\cos 1 \tag{17}$$

The exact solution of (16) subject to (17) in the case

$$g(t) = -t^{1-r}\cos t - (2-r)t^{-r}\sin t$$

is

$$u(t) = t^{1-r}\cos t \tag{18}$$

To solve (16)-(17) by means of HAM [7], we choose the initial boundary approximation

$$u_0(t) = t^{1-r}\cos 1 \tag{19}$$

and the linear operator

$$\mathcal{L}[\phi(t;q)] = \frac{\partial^2\phi(t;q)}{\partial t^2} + \frac{r}{t}\frac{\partial\phi(t;q)}{\partial t} \tag{20}$$

Eq. (16) suggests that we define a nonlinear operator

$$\mathcal{N}[\phi(t;q)] = \frac{\partial^2\phi(t;q)}{\partial t^2} + \frac{r}{t}\frac{\partial\phi(t;q)}{\partial t} \tag{21}$$

Using the above definition, we construct the zeroth-order deformation equation as in (7) and the mth-order deformation equation for $m \geq 1$ as in (9), where

$$R_m(\vec{u}_{m-1}) = u''_{m-1} + \frac{r}{t}u'_{m-1} - (1-\chi_m)g(t)$$

In this example we can see that

$$u_m(t) = \chi_m u_{m-1}(t) + \hbar\int_0^t \tau^{-r}\int_0^\tau \mu^r R_m(\vec{u}_{m-1})d\mu d\tau + c_0 + c_1 t^2$$

where the integral constants c_0 and c_1 are determined by the initial conditions (17). We now successively obtain

$$u_1(t) = \hbar t^{1-r}(\cos 1 - \cos t) \tag{22}$$

$$u_m(t) = \hbar(1+\hbar)^{m-1} t^{1-r}(\cos 1 - \cos t), \geq 2 \tag{23}$$

Note that the homotopy-series solution (11) contains the convergence-control parameters $\hbar$, which influences the convergence of the homotopy-series (11). Thus, mathematically, the series solution is dependent upon $\hbar$ also physically. Hence, the homotopy-series must converge to the same result for all corresponding values of $\hbar$ which ensures the convergence. As mentioned by Liao [7, 8], the admissible values of $\hbar$ for which the homotopy-series converges can be determined by plotting the so-called $\hbar$-curves or by plotting the residual error verses $\hbar$. Figure 1, the $\hbar$-curve of this example for r = -2.5. We can see that the approximate region for the convergence of the homotopy-series is about that $-1.5 \leq \hbar \leq -0.5$.

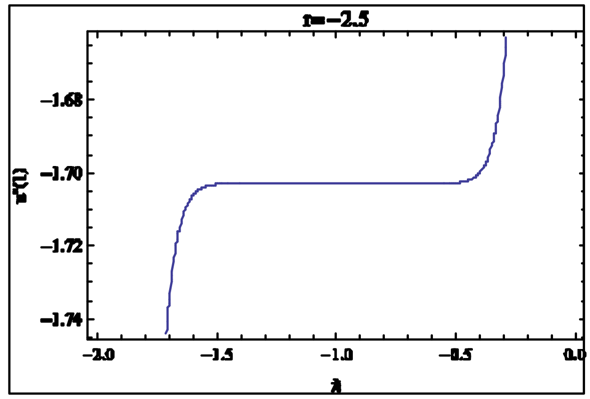

Fig. 1: The $\hbar$-curve of $u''(1)$ when r = -2.5 given by (16): 7th-order approximation of $u''(1)$

Let $\delta_m(t)$ denotes the residual error of the mth-order homotopy-series approximation

$$\Delta_m = \int_0^1 \delta_m^2(t)dt$$

The curve of $\Delta_m \sim \hbar$, is straight forward to find a region of $\hbar$ in which Δ_m decreases to zero as the order of approximation increases. In this way, we can get the best value of $\hbar$ corresponding to the minimum of the residual error of the original governing equation.
In this example we can find the

$$\Delta_m = C_r(1+\hbar)^{2m}$$

where C_r depends on r. Hence the best value for $\hbar$ in this example is -1. Then the series solutions expression by HAM is

$$u(t) = t^{1-r}\cos t \tag{24}$$

It is the exact solution of (18).

Remark 1. Here, we choose $u_0(t) = t^{1-r}\cos 1$, which is very clear from the conditions (17) indeed we want to have $\mathcal{L}[u_0(t)] = 0$.

Example 2: Now, we deal with the following nonlinear singular BVPs [3],

$$u''(t) - \frac{1}{t}u'(t) = f(t)u^5(t) \tag{25}$$

subject to the boundary conditions

$$u(0) = 1, u'(1) = -0.216506 \tag{26}$$

The exact solution of (24) subject to (25) in the case $f(t)=\frac{t^2}{3}$ is

$$u(t)=\frac{1}{\sqrt{1+\frac{t^2}{3}}} \tag{27}$$

Again, to solve (25)-(26) by means of HAM, we choose the initial boundary approximation

$$u_0(t)=1-0.108253t^2 \tag{28}$$

and the linear operator

$$\mathcal{L}[\phi(t;q)]=\frac{\partial^2\phi(t;q)}{\partial t^2}-\frac{1}{t}\frac{\partial\phi(t;q)}{\partial t} \tag{29}$$

Eq. (25) suggests that we define a nonlinear operator

$$\mathcal{N}[\phi(t;q)]=\frac{\partial^2\phi(t;q)}{\partial t^2}-\frac{1}{t}\frac{\partial\phi(t;q)}{\partial t}-f(t)\phi^5(t;q) \tag{30}$$

Using the above definition, we construct the *zeroth-order deformation equation* as in (7) and the *mth-order deformation equation* for $m\geq 1$ is as in (9) with the boundary conditions

$$u_m(0)=0,\quad u'_m(1)=0 \tag{31}$$

where

$$R_m(\vec{u}_{m-1})=u''_{m-1}-\frac{1}{t}u'_{m-1}-\frac{t^2}{3}\sum_{r_1=0}^{m-1}u_{m-r_1-1}\sum_{r_2=0}^{r_1}u_{r_1-r_2}\sum_{r_3=0}^{r_2}u_{r_2-r_3}\sum_{r_4=0}^{r_3}u_{r_3-r_4}u_{r_4}$$

We now successively obtain

$$\begin{aligned}u_1(t)=\hbar(&0.0637828t^2-0.0416667t^4+0.00751757t^6\\&-0.000813799t^8+0.0000528577t^{10}\\&-1.9073362659705669\times 10^{-612}\\&+2.9496410400015968\times 10^{-814})\end{aligned} \tag{32}$$

etc. Figure 2, the $\hbar$-curve of this example for 10th-order approximation of $u'''(0)$. We can see that the approximate region for the convergence of the homotopy-series is about that $-1.3\leq\hbar\leq -0.75$. Figure 3, the curve of $\Delta_m\sim\hbar$, we can get the best value of $\hbar$ corresponding to the minimum of the residual error of the original governing equation. From this figure one see that $\hbar=-0.88$ can be chosen. Hence, we obtain the following

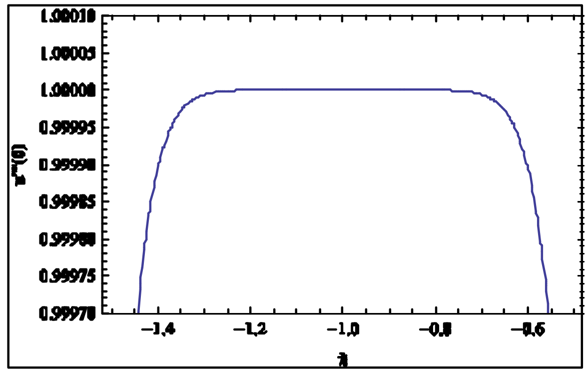

Fig. 2: The $\hbar$-curve of $u'''(0)$ given by (25): 10th-order approximation of $u''''(0)$

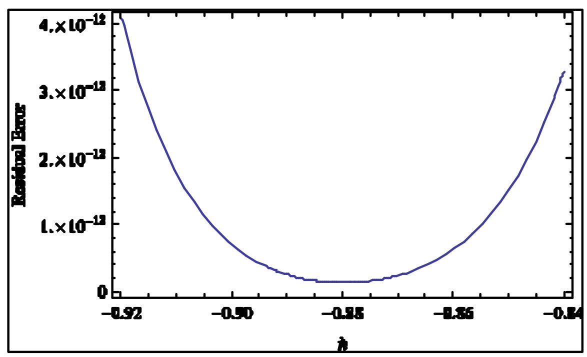

Fig. 3: The residual error curve of example 2

$$\begin{aligned}u(t)\approx\ &1-0.166667x^2+0.0416667x^4-0.0115741x^6\\&+0.00337577x^8-0.00101273x^{10}\\&+0.000309445x^{12}-0.00009578x^{14}\\&+0.0000299306x^{16}-\cdots\end{aligned} \tag{33}$$

which converges to the closed-form solution (27).

Example 3: Finally, we consider the nonlinear singular BVPs [3],

$$u'''(t)-\frac{2}{t}u''(t)-u^3(t)=g(t) \tag{34}$$

subject to the boundary conditions

$$u(0)=0,\quad u'(0)=0,\quad u(1)=10.8731 \tag{35}$$

The exact solution of (34) subject to (35) in the case

$$g(t)=7t^2e^t+6te^t-6e^t-t^9e^{3t}+t^3e^t$$

is

$$u(t)=t^3e^t \tag{36}$$

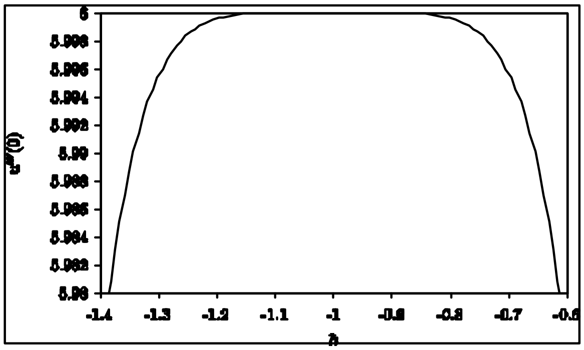

Fig. 4: The $\hbar$-curve of $u'''(0)$ given by (34): 7th-order approximation of $u'''(0)$

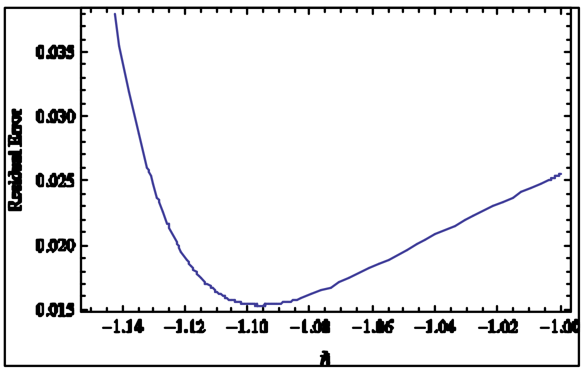

Fig. 5: The residual error curve of examp le 3

In [3] this example is considered by modified Adomian decomposition method. HAM also cannot produce a reasonable solution. Before we apply the MHAM [23], we approximate g(t) by the 10-term Taylor series expansion,

$$g(t) \approx -6 + 10t^2 + 10t^3 + \frac{21}{4}t^4 + \frac{28}{15}t^5 + \frac{1}{2}t^6 + \frac{3}{28}t^7 + \frac{11}{576}t^8 - \frac{3769}{3780}t^9 - \frac{100787}{33600}t^{10}$$

We choose the initial boundary approximation

$$u_0(t) = \frac{10.8731}{4}t^4 \tag{37}$$

and the linear operator

$$\mathcal{L}[\phi(t;q)] = \frac{\partial^3\phi(t;q)}{\partial t^3} - \frac{2}{t}\frac{\partial^2\phi(t;q)}{\partial t^2} \tag{38}$$

Now by using (12) divide g(t) with respect to q as:

$$k_0(t) = -6q^0,\quad k_1(t) = 10t^2q^1,\quad \ldots,$$
$$k_9(t) = -\frac{100787}{33600}t^{10}q^9,\quad \ldots.$$

Using the above definition, we construct the zeroth-order deformation equation as in (13) and the mth-order deformation equation for $m \geq 1$ is as in (14) with the boundary conditions

$$u_m(0)=0,\quad u'_m(0)=0,\quad u'_m(1)=0 \tag{39}$$

where

$$\mathcal{R}_m(\vec{u}_{m-1}) = u'''_{m-1} - \frac{2}{t}u''_{m-1} - \sum_{r_1=0}^{m-1} u_{m-r_1-1} \sum_{r_2=0}^{r_2} u_{r_1-r_2}u_{r_2} - k_m(t)$$

Figure 4, the $\hbar$-curve of this example for 7th-order approximation of $u'''(0)$. We can see that the approximate region for the convergence of the homotopy-series is about that $-1.15 \leq \hbar \leq -0.85$. Figure 5, the curve of residual error, we can get the best value of $\hbar$ as the pervious example. From this figure one see that $\hbar \approx -1.1$ can be chosen. Hence, we obtain the following

$$u(t) \approx x^3 + 0.999468x^4 + 0.5x^5 + 0.166667x^6 + 0.0416667x^7 + 0.0083333x^8 + 0.00138889x^9 + 0.000198413x^{10} + \ldots.$$

which converges to the closed-form solution (36).

CONCLUSIONS

In this paper, the HAM and MHAM were applied to solve a class of singular higher-order two-point BVPs. The HAM and MHAM provide us with a convenient way of controlling the convergence of approximation series. It was shown that HAM and MHAM are capable of yielding exact/approximate solutions of the BVPs.

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