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Minimal Immersions In Sol Space

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Abstract: In this paper, we give a setting for constructing a Weierstrass representation formula for simply connected minimal surfaces in the Sol space. We derive the Weierstrass representation for surfaces in the three-dimensional Lie group Sol space and establish the generating equations for minimal surfaces in the group Sol space.

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INTRODUCTION

In the last decade the study of the geometry of surfaces in the 3-dimensional Thurston geometries has grown considerably. One reason is that these spaces can be endowed with a complete metric with a large isometry group; another, more recent, is the announced proof of Thurston geometric conjecture, which ensures the dominant role of this spaces among the 3-dimensional geometries.

Analytic methods to study surfaces and their properties are of great interest both in mathematics and in physics. A classical example of such an approach is given by the Weierstrass representation for minimal surfaces [1]. This representation allows us to construct any minimal surface in the three-dimensional Euclidean space R^3 via two holomorphic functions. It is the most powerful tool for the analysis of minimal surfaces.

Weierstrass representations are very useful and suitable tools for the systematic study of minimal surfaces immersed in n-dimensional spaces [2]. This subject has a long and rich history. It has been extensively investigated since the initial works of Weierstrass [3]. In the literature there exists a great number of applications of the Weierstrass representation to various domains of Mathematics, Physics, Chemistry and Biology. In particular in such areas as quantum field theory [4], statistical physics [5], chemical physics, fluid dynamics and membranes [6, 7], minimal surfaces play an essential role.

In this paper, we give a setting for constructing a Weierstrass representation formula for simply connected minimal surfaces in the Sol space. We derive the Weierstrass representation for surfaces in the three-dimensional Lie group Sol space and establish the generating equations for minimal surfaces in the group Sol space. As a consequence, we shall give a Weierstrass-type representation formula for minimal surfaces in the 3-dimensional Lie group Sol.

Riemannian Structure of Sol Space: Sol space, one of Thurston's eight 3-dimensional geometries, can be viewed as R^3 provided with Riemannian metric

$$g_{sol} = ds^2 = e^{2x3}\, dx_1^{\ 2} + e^{-2t}\, dx^2_{\ 2} + dx^2_{\ 3} \tag{2.1}$$

Where (x_1,x_2,x_3) are the standard coordinates in R^3. Note that the Sol metric can also be written as:

$$ds^2 = \sum_{i=1}^{3} ?^{\,i} \otimes ?^{\,i}, \tag{2.2}$$

where

$$?^{\,1} = e^{x_3} dx_1,\ ?^{\,2} = e^{-x_3} dx_2, s\ ?^{\,3} = dx_3, \tag{2.3}$$

and the orthonormal basis dual to the 1-forms is

$$\mathbf{e}_1 = e^{-x_3}\frac{\partial}{\partial x_1},\ \mathbf{e}_2 = e^{x_3}\frac{\partial}{\partial x_2},\ \mathbf{e}_3 = \frac{\partial}{\partial x_3}. \tag{2.4}$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g_{sol}, defined above the following is true:*

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$$\nabla = \begin{pmatrix} -\mathbf{e}_3 & 0 & \mathbf{e}_1 \\ 0 & \mathbf{e}_3 & -\mathbf{e}_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.5}$$

where the (i, j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ *for our basis*

$$\{e_k,\ k = 1,2,3,\}\ \{e_1,e_2,e_3\}$$

Lie brackets can be easily computed as:

$$[e_1,e_2] = 0,\ [e_2,e_3] = -e_2,\ [e_1,e_3] = e_1. \tag{2.6}$$

Then, we write the Kozul formula for the Levi--Civita connection is:

$$2g(\nabla_{\mathbf{e}_i}\mathbf{e}_j,\mathbf{e}_k) = L_{ij}^k.$$

From (2.4), we get

$$L_{21}^3 = 2, L_{23}^1 = -2, L_{12}^3 = 2, L_{31}^2 = -2, L_{32}^1 = -2 \tag{2.4}$$

Weierstrass Representation Formula on Sol Space: $\Sigma \subset (R^3, g_{sol})$ be $\wp : \Sigma \to \left(R^3, g_{sol}\right)$ a surface and a smooth map. The pull-back bundle $\wp^*(T\left(R^3, g_{sol}\right))$ has a metric and compatible connection, the pull-back connection, induced by he Riemannian metric and the Levi--Civita connection of (R^3, g_{sol}). Consider the complexified bundle $E = \wp^*(T\left(R^3, g_{sol}\right)) \otimes C$.

Let (u, v) be local coordinates on Σ, and $z = u + iv$ the (local) complex parameter and set, as usual,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right),\quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right). \tag{3.1}$$

Let

$$\frac{\partial \wp}{\partial u}|_p = \wp_{*p}\left(\frac{\partial}{\partial u}|_p\right),\quad \frac{\partial \wp}{\partial v}|_p = \wp_{*p}\left(\frac{\partial}{\partial v}|_p\right), \tag{3.2}$$

and

$$\phi = \wp_z = \frac{\partial \wp}{\partial z} = \frac{1}{2}\left(\frac{\partial \wp}{\partial u} - i\frac{\partial \wp}{\partial v}\right). \tag{3.3}$$

Let now $\wp : \Sigma \to \left(R^3, g_{sol}\right)$ be a conformal immersion and $z = u + iv$ a local conformal parameter. Then, the induced metric is

$$ds^2 = \lambda^2(du^2 - dv^2) = \lambda^2\,|dz|^2, \tag{3.4}$$

and the Beltrami--Laplace operator on (R^3, g_{sol}), with respect to the induced metric, is given by

$$\Delta = \lambda^{-2}\left(\frac{\partial}{\partial u}\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\frac{\partial}{\partial v}\right). \tag{3.5}$$

We recall that a map $\wp : \Sigma \to \left(R^3, g_{sol}\right)$ is harmonic if its tension field

$$\tau(\wp) = \mathrm{trace}\nabla d\wp = 0. \tag{3.6}$$

Let $\{x_1,x,x_3,\}$ be a system of local coordinates in a neighborhood U of M such that $U \cap \wp(\Sigma) \neq \varnothing$. Then, in an open set $G \subset \Sigma$

$$\phi = \sum_{j=1}^{3}\phi_j\frac{\partial}{\partial x_j}, \tag{3.7}$$

for some complex-valued functions ϕ_j defined on G. With respect to the local decomposition of ϕ, the tension field can be written as

$$\tau(\wp) = \sum_i\left\{\Delta\wp_i + 4\lambda^{-2}\sum_{j,k=1}^{n}\Gamma_{jk}^i\frac{\partial\wp_j}{\partial\bar{z}}\frac{\partial\wp_k}{\partial z}\right\}\frac{\partial}{\partial x_i}, \tag{3.8}$$

Where Γ_{jk}^i are the Christoffel symbols of (R^3, g_{sol}).

From (3.3), we have

$$\tau(\wp) = 4\lambda^{-2}\sum_i\left\{\frac{\partial\phi_i}{\partial\bar{z}} + \sum_{j,k=1}^{n}\Gamma_{jk}^i\bar{\phi}_j\phi_k\right\}\frac{\partial}{\partial x_i}.$$

The section ϕ is holomorphic if and only if

$$\nabla_{\frac{\partial}{\partial\bar{z}}}\left(\sum_{i=1}^{3}\phi_i\frac{\partial}{\partial x_i}\right) = \sum_i\left\{\frac{\partial\phi_i}{\partial\bar{z}}\frac{\partial}{\partial x_i} + \phi_i\nabla_{\frac{\partial\zeta}{\partial\bar{z}}}\frac{\partial}{\partial x_i}\right\}.$$

Using (3.3), we get

$$\nabla_{\frac{\partial}{\partial\bar{z}}}\left(\sum_{i=1}^{3}\phi_i\frac{\partial}{\partial x_i}\right) = \sum_i\left\{\frac{\partial\phi_i}{\partial\bar{z}}\frac{\partial}{\partial x_i} + \phi_i\nabla_{\sum_j\bar{\phi}_j\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_i}\right\}.$$

Making necessary calculations, we obtain

$$\nabla_{\frac{\partial}{\partial\bar{z}}}\left(\sum_{i=1}^{3}\phi_i\frac{\partial}{\partial x_i}\right) = \sum_i\left\{\frac{\partial\phi_i}{\partial\bar{z}} + \sum_{j,k}\Gamma_{jk}^i\bar{\phi}_j\phi_k\right\}\frac{\partial}{\partial x_i} = 0.$$

Thus, ϕ is holomorphic if and only if

$$\frac{\partial \phi_i}{\partial \bar{z}} + \sum_{j,k} \Gamma^i{}_{jk} \bar{\phi}_j \phi_k = 0, \; i = 1,2,3. \tag{3.9}$$

Theorem 3.1: *(Weierstrass representation) Let* (R^3, g_{sol}) *be the group of rigid motions of Euclidean 2-space and* $\{x_1,x_2,x_3,\}$ *local coordinates. Let* ϕ, j = 1,2,3 *be complex-valued functions in an open simply connected domain G* $\subset C$ *which are solutions of (3.9). Then, the map*

$$\wp_j(u,v) = 2Re\left(\int_{z_0}^{z} \phi_j dz\right) \tag{3.10}$$

is well defined and defines a minimal conformal immersion if and only if the following conditions are satisfied:

$$\sum_{j,k=1}^{3} g_{ij}\phi_j\overline{\phi_k} \neq 0 \text{and} \sum_{j,k=1}^{3} g_{ij}\phi_j\phi_k = 0 .$$

Let us expand Υ with respect to this basis to obtain

$$\Upsilon = \sum_{k=1}^{3} \psi_k \mathbf{e}_k. \tag{3.11}$$

Setting

$$\phi = \sum_i \phi i \frac{\partial}{\partial x_i} = \sum_i \psi_i e_i, \tag{3.12}$$

for some complex functions ϕ_i, $\psi_i : G \subset C$. Moreover, there exists an invertible matrix $A = (A_{ij})$, with function entries $A_{ij} : \wp(G) \cap U \to R$, , i, j = 1,2,3 such that

$$\phi_i = \sum_j A_{ij}\psi_j. \tag{3.13}$$

Using the expression of ϕ, the section ϕ is holomorphic if and only if

$$\frac{\partial \psi_i}{\partial \bar{z}} + \frac{1}{4}\sum_{j,k} L^i{}_{jk} \bar{\psi}_j \psi_k = 0, i = 1,2,3. \tag{3.14}$$

Theorem 3.2: *Let* ψ_i, j = 1,2,3, *be complex-valued functions defined in a open simply connected set* $G \subset C$, *such that the following conditions are satisfied:*

i. $|\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 \neq 0$,
ii. $\psi_1^2 + \psi_2^2 + \psi_3^2 = 0$,
iii. ψ_j *are solutions of (3.21).*

Then, the map $\wp : G \to \left(R^3, g_{sol}\right)$ *defined by*

$$\wp_i(u,v) = 2Re\left(\int_{z_0}^{z} \sum_j A_{ij}\psi_j dz\right) \tag{3.15}$$

is a conformal minimal immersion.

Proof: By Theorem 3.1 we see that $\wp$ is a harmonic map if and only if $\wp$ satisfy (3.15). Then, the map $\wp$ is a conformal minimal immersion.

Since the parameter z is conformal, we have

$$\langle \Upsilon, \Upsilon \rangle = 0, \tag{3.16}$$

which is rewritten as

$$\psi_1^2 + \psi_2^2 + \psi_2^2 = 0. \tag{3.17}$$

Case I: From (3.17), we have

$$\frac{\partial \psi_2}{\partial \bar{z}} = \bar{\psi}_2 \psi_3, \tag{3.18}$$

Which suggests the definition of two new complex functions

$$B := \sqrt{\frac{1}{2}(\psi_1 - i\psi_2)}, \;\; H := \sqrt{-\frac{1}{2}(\psi_1 + i\psi_2)}. \tag{3.19}$$

The functions G and H are single-valued complex functions which, for suitably chosen square roots, satisfy

$$\begin{aligned} \psi_1 &= B_2 + H_2 \\ \psi_2 &= i(B_2 - H_2) \\ \psi_3 &= 2BH. \end{aligned} \tag{3.20}$$

Lemma 3.3: *The section* $\phi = \psi_1 e_1 + \psi_2 e_2 + \psi_3 e_3$ *is holomorphic if and only if*

$$\begin{gathered} \frac{\partial \psi_1}{\partial \bar{z}} + \frac{i}{2}\left(|B|^2 + |H|^2\right)\left(|B|^2 - |H|^2\right) = 0, \\ \frac{\partial \psi_2}{\partial \bar{z}} + \frac{i}{2}\left(|B|^2 - |H|^2\right)G\overline{H} = 0, \\ \frac{\partial \psi_3}{\partial \bar{z}} + 2B^2\overline{H}^2 = 0. \end{gathered} \tag{3.21}$$

Proof: Using (3.14) and (3.18), we have (3.21).

Theorem 3.4: *Let B and H be complex-valued functions defined in a simply connected domain $G \subset C$ such that:*

- *B and H are not identically zero,*
- *B and H are solutions of (3.21).*

Then the map $\wp : G \to \left(\mathrm{R}^3, g_{sol}\right)$, *defined by*

$$\wp_1(u,v) = 2Re\left(\int_{z_0}^{z}\left(B^2 - H^2\right)e^{-x_3}dz\right),$$

$$\wp_2(u,v) = 2Re\left(\int_{z_0}^{z}\left(B^2 - H^2\right)e^{x_3}dz\right), \tag{3.22}$$

$$\wp_2(u,v) = 4Re\left(\int_{z_0}^{z}BHdz\right).$$

Corollary 3.5: *If the section $\phi = \psi_1 e_1 + \psi_2 e_2 + \psi_3 e_3$ is holomorphic and $B = e^{i\theta}$, then*

$$\left|\psi_1\right|^2 - \left|\psi_2\right|^2 = i\left(\psi_1\overline{\psi}_2 + \overline{\psi}_1\psi_2\right) \text{and} \psi_1^2 = \psi_2^2. \tag{3.23}$$

Proof: Substituting $B = e^{i\theta}$ into (3.21), we get

$$B\frac{\partial B}{\partial \overline{z}} - H\frac{\partial H}{\partial \overline{z}} = 0,$$
$$B\frac{\partial B}{\partial \overline{z}} + H\frac{\partial H}{\partial \overline{z}} = 0.$$

The first and adding the second we get

$$\frac{\partial B}{\partial \overline{z}} = 0,\ \frac{\partial H}{\partial \overline{z}} = 0.$$

Multiplying the first B and the second H adding, we get

$$\frac{\partial}{\partial \overline{z}}(2BH) = 0. \tag{3.24}$$

From (3.18), we have

$$\frac{\partial \psi_3}{\partial \overline{z}} = 0. \tag{3.25}$$

Substituting (3.25) into (3.21), we get $B^2\overline{H}^2 = 0.$ From (3.18), we obtain (3.23). This proves the claim.

Corollary 3.6: *If the section $\phi = \psi_1 e_1 + \psi_2 e_2 + \psi_3 e_3$ is holomorphic and $B = e^{i\theta}$, then*

$$\frac{\partial \psi_1}{\partial \overline{z}} = 2e^{-i\theta}\frac{\partial \psi_2}{\partial \overline{z}}. \tag{3.26}$$

Proof: From (3.18), we obtain

$$\frac{\dfrac{\partial \psi_1}{\partial \overline{z}}}{\dfrac{\partial \psi_2}{\partial \overline{z}}} = \frac{|B|^2 + |H|^2}{G\overline{H}}, \tag{3.27}$$

$$B\overline{H}\frac{\partial \psi_1}{\partial \overline{z}} = (|B|^2 + |H|^2)\frac{\partial \psi_2}{\partial \overline{z}}. \tag{3.28}$$

Substituting $B = e^{i\theta}$ into (3.28), we get (3.26).

Case II: From (3.17), we have

$$\psi_1 = \Re\cos\Im,\ \psi_2 = \Re\sin\Im,\ \psi_3 = \Re \tag{3.29}$$

Which suggests the definition of two new complex functions

$$\Im = \arctan\frac{\psi_2}{\psi_1} \text{and} \Re = \psi_1^2 + \psi_2^2. \tag{3.30}$$

Lemma 3.7: *If Υ satisfies the equation (3.14), then*

$$\Re_{\overline{z}}\cos\Im - \Im_{\overline{z}}\Re\sin\Im = -|\Re|\overline{\cos\Im}, \tag{3.31}$$

$$\Re_{\overline{z}}\sin\Im + \Im_{\overline{z}}\Re\cos\Im = |\Re|\overline{\sin\Im}, \tag{3.32}$$

$$\Re_{\overline{z}} = -|\Re|\left(|\cos\Im| - |\sin\Im|\right). \tag{3.33}$$

Proof: Using (2.5) and (3.14), we have

$$\frac{\partial \psi_1}{\partial \overline{z}} = -\overline{\psi}_1\psi_3,$$
$$\frac{\partial \psi_2}{\partial \overline{z}} = \overline{\psi}_2\psi_3, \tag{3.34}$$

$$\frac{\partial \psi_3}{\partial \overline{z}} = |\psi_1|^2 - |\psi_2|^2.$$

Substituting (3.29) into (3.34), we have (3.31)-(3.33).

Corollary 3.8:

$$\frac{\partial \psi_1}{\partial \bar{z}} \cos \Im + \frac{\partial \psi_2}{\partial \bar{z}} \sin \Im = -\frac{\partial \psi_3}{\partial \bar{z}}. \tag{3.35}$$

Theorem 3.9: *Let $\Re$ and $\Im$ be complex-valued functions defined in a simply connected domain $G \subset C$. Then the map* $\wp : G \to \left(\mathbb{R}^3, g_{sol}\right)$, *defined by*

$$\begin{aligned} \wp_1(u,v) &= Re\left(\left(e^{-x_3} \Re \cos \Im\right) dz\right), \\ \wp_2(u,v) &= Re\left(\int_{z_0}^{z} \left(e^{x_3} \Re \sin \Im\right) dz\right), \\ \wp_3(u,v) &= Re\left(\int_{z_0}^{z} (\Re) dz\right) \end{aligned} \tag{3.36}$$

is a conformal minimal immersion.

Proof: Using (3.10) and (3.32), we get

$$\phi_1 = e^{-x_3} \Re \cos \Im, \phi_2 = e^{x_3} \Re \sin \Im, \phi_3 = \Re.$$

Using Theorem 3.2 $\wp : G \to \left(\mathbb{R}^3, g_{sol}\right)$ is a conformal minimal immersion.

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