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A Novel Homotopy Perturbation Transform Algorithm for Linear and Nonlinear System of Partial Differential Equations

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Abstract: In this paper, a Homotopy Perturbation Transform Algorithm (HPTA) which is based on the Homotopy Perturbation Method (HPM) is introduced for the approximate solution of the linear and nonlinear system of partial differential equations. Illustrative examples are included to demonstrate the high accuracy and fast convergence of proposed new algorithm.

Key words: Homotopy perturbation transform algorithm . homotopy perturbation method

INTRODUCTION

System of partial differential equations have attracted much attention in a variety of applied sciences. The general ideas and the essential features of these system are of wide applicability. These system were formally derived to describe wave propagation, to control the shallow water waves and to examine the chemical reaction-diffusion model of Brusselator. The method of characteristics, the Riemann invariants, Adomian decomposition method [1], Homotopy perturbation method [2-5], Homotopy analysis method [6, 7] and Laplace decomposition method [8-10], were the commonly used methods. In this work, we will use Homotopy perturbation transform algorithm introduced by Yasir *et al*. [11, 12]. Majid *et al*. [13] solved exponential stretching sheet equation with the help HPTA on semi infinite domian. This new algorithm basically illustrates how two powerful algorithms, homotopy perturbation method and Laplace decomposition method can be combined and used to approximate the solutions of the nonlinear partial differential equations by manipulating the homotopy perturbation method.

HOMOTOPY PERTURBATION TRANSFORM ALGORITHM

In this section, we present a homotopy perturbation transform algorithm for solving system of partial differential equations written in an operator form

$$\begin{cases} L_t u + R_1(u,v) + N_1(u,v) = g_1 \\ L_t v + R_2(u,v) + N_2(u,v) = g_2 \end{cases} \quad (2.1)$$

with the initial conditions

$$\begin{cases} u(x,0) = f_1(x) \\ v(x,0) = f_2(x) \end{cases} \quad (2.2)$$

where L_t is consider as a first-order partial differential operator, R_1, R_2 and N_1, N_2 are linear and nonlinear operators and g_1 and g_2 are source terms. The method consists of first applying the Laplace transform to both sides of equations in system (2.1) and then by using initial conditions (2.2), we have

$$£[L_t u] + £[R_1(u,v)] + £[N_1(u,v)] = £[g_1] \quad (2.3)$$

$$£[L_t v] + £[R_2(u,v)] + £[N_2(u,v)] = £[g_2] \quad (2.4)$$

Using the differential property of Laplace transform and initial conditions, we have

$$s£[u(x,t)] - u(x,0) + £[R_1(u,v)] + £[N_1(u,v)] = £[g_1(x,t)] \quad (2.5)$$

$$s£[v(x,t)] - v(x,0) + £[R_2(u,v)] + £[N_2(u,v)] = £[g_2(x,t)] \quad (2.6)$$

On Simplifying

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$$£[u]=\frac{f_1(x)}{s}+\frac{1}{s}£[g_1]-\frac{1}{s}£[R_1(u,v)]-\frac{1}{s}£[N_1(u,v)] \quad (2.7)$$

$$£[v]=\frac{f_2(x)}{s}+\frac{1}{s}£[g_2]-\frac{1}{s}£[R_2(u,v)]-\frac{1}{s}£[N_2(u,v)] \quad (2.8)$$

Applying inverse Laplace transform on both sides of Eqs. (2.7)-(2.8), we get

$$u=F_1(x)-£^{-1}\left[\frac{1}{s}\left[£[N_1(u,v)]+£[R_1(u,v)]\right]\right] \quad (2.9)$$

$$v=F_2(x)-£^{-1}\left[\frac{1}{s}\left[£[N_2(u,v)]+£[R_2(u,v)]\right]\right] \quad (2.10)$$

where $F_1(x)$ and $F_2(x)$ represents the terms arising from source terms and prescribe initial conditions. According to standard homotopy perturbation method the solution u and v can be expanded into infinite series as

$$u=\sum_{m=0}^{\infty}p^m u_m, \qquad v=\sum_{m=0}^{\infty}p^m v_m \quad (2.11)$$

where $p\in[0,1]$ is an embedding parameter. Also the nonlinear term N_1 and N_2 can be written as

$$N_1(u,v)=\sum_{m=0}^{\infty}p^m H_{1m}(u,v) \quad (2.12)$$

$$N_2(u,v)=\sum_{m=0}^{\infty}p^m H_{2m}(u,v)$$

where H_{1m} and H_{2m} are the He's polynomials [11]. By substituting Eqs. (2.11) and (2.12) in Eqs. (2.9)- (2.10), the solutions can be written as

$$\sum_{m=0}^{\infty}p^m u_m=F_1(x)-p\left(£^{-1}\left[\frac{1}{s}\left[£[H_{1m}]+£[R_1(u,v)]\right]\right]\right) \quad (2.13)$$

$$\sum_{m=0}^{\infty}p^m v_m=F_2(x)-p\left(£^{-1}\left[\frac{1}{s}\left[£[H_{2m}]+£[R_2(u,v)]\right]\right]\right) \quad (2.14)$$

In Eqs. (2.13)-(2.14), H_{1m}, H_{2m} are He's polynomials can be generated by several means. Here we used the following recursive formulation:

$$H_m(u_0,\ldots,u_m)=\frac{1}{m!}\frac{\partial^m}{\partial p^m}\left[N\left(\sum_{i=0}^{\infty}p^i u_i\right)\right]_{p=0}, \quad m=0,1,2,\ldots \quad (2.15)$$

Equating the terms with identical powers in p in Eqs. (2.13)-(2.14), we obtained the following approximations

$$\begin{aligned} p^0 &: u_0=F_1(x)\\ p^1 &: u_1=-£^{-1}\left[\frac{1}{s}\left[£[H_{10}]+£[R_1(u_0,v_0)]\right]\right]\\ p^2 &: u_2=-£^{-1}\left[\frac{1}{s}\left[£[H_{11}]+£[R_1(u_1,v_1)]\right]\right]\\ &\vdots \end{aligned} \quad (2.16)$$

Similarly

$$\begin{aligned} p^0 &: v_0=F_2(x)\\ p^1 &: v_1=-£^{-1}\left[\frac{1}{s}\left[£[H_{20}]+£[R_2(u_0,v_0)]\right]\right]\\ p^2 &: v_2=-£^{-1}\left[\frac{1}{s}\left[£[H_{21}]+£[R_2(u_1,v_1)]\right]\right]\\ &\vdots \end{aligned} \quad (2.17)$$

The best approximations for the solutions are

$$u=\lim_{p\to 1}u_m=u_0+u_1+u_2+\ldots \quad (2.18)$$

$$v=\lim_{p\to 1}v_m=v_0+v_1+v_2+\ldots \quad (2.19)$$

This method does not resort to linearization or assumptions of weak nonlinearity, the solution generated in the form of general solution and it is more realistic compared to the method of simplifying the physical problems.

APPLICATIONS

In this section, we use the HPTA to solve homogeneous and inhomogeneous linear system of partial differential equations and homogeneous and inhomogeneous nonlinear system of partial differential equations.

The homogeneous linear system: Consider the homogeneous linear system of PDEs

$$u_t+v_x-(u+v)=0 \quad (3.1)$$

$$v_t+u_x-(u+v)=0 \quad (3.2)$$

with initial conditions

$$u(x,0)=\sinh(x), \quad v(x,0)=\cosh(x) \quad (3.3)$$

Applying Laplace transform algorithm we have

$$su(x,s)-u(x,0)=-£[v_x]+£[(u+v)] \quad (3.4)$$

$$sv(x,s)-v(x,0)=-£[u_x]+£[(u+v)] \quad (3.5)$$

$$u(x,s) = \frac{u(x,0)}{s} - \frac{1}{s}£\left[v_x - (u+v)\right] \quad (3.6)$$

$$v(x,s) = \frac{v(x,0)}{s} - \frac{1}{s}£\left[u_x - (u+v)\right] \quad (3.7)$$

Using given initial condition Eqs. (3.6)-(3.7), becomes

$$u(x,s) = \frac{\sinh(x)}{s} - \frac{1}{s}£\left[v_x - (u+v)\right] \quad (3.8)$$

$$v(x,s) = \frac{\cosh(x)}{s} - \frac{1}{s}£\left[u_x - (u+v)\right] \quad (3.9)$$

Applying inverse Laplace transform to Eqs. (3.8)-(3.9), we get

$$u(x,t) = \sinh(x) - £^{-1}\left[\frac{1}{s}£\left[v_x - (u+v)\right]\right] \quad (3.10)$$

$$v(x,t) = \cosh(x) - £^{-1}\left[\frac{1}{s}£\left[u_x - (u+v)\right]\right] \quad (3.11)$$

The Homotopy Perturbation Transform Algorithm (HPTA) assumes a series solutions of the functions u(x,t) and v(x,t) is given by

$$u = \sum_{m=0}^{\infty} p^m u_m(x,t),\ \ v = \sum_{m=0}^{\infty} p^m v_m(x,t) \quad (3.12)$$

Using Eq. (3.12) into Eqs. (3.10)-(3.11), we get

$$\sum_{m=0}^{\infty} p^m u_m(x,t) = \sinh(x) - p\left(£^{-1}\left[\frac{1}{s}£\left[\begin{array}{c}\left(\sum_{m=0}^{\infty} p^m v_m(x,t)\right)_x - \\ \left(\begin{array}{c}\sum_{m=0}^{\infty} p^m v_m(x,t) + \\ \sum_{m=0}^{\infty} p^m v_m(x,t)\end{array}\right)\end{array}\right]\right]\right) \quad (3.13)$$

$$\sum_{m=0}^{\infty} p^m v_m(x,t) = \cosh(x) - p\left(£^{-1}\left[\frac{1}{s}£\left[\begin{array}{c}\left(\sum_{m=0}^{\infty} p^m u_m(x,t)\right)_x - \\ \left(\begin{array}{c}\sum_{m=0}^{\infty} p^m v_m(x,t) + \\ \sum_{m=0}^{\infty} p^m v_m(x,t)\end{array}\right)\end{array}\right]\right]\right) \quad (3.14)$$

From Eqs. (3.9)-(3.12), comparing like powers of p yields

$$p^0 : \begin{cases} u_0(x,t) = \sinh(x) \\ v_0(x,t) = \cosh(x) \end{cases} \quad (3.15)$$

$$p^1 : \begin{cases} u_1(x,t) = t\sinh(x) \\ v_1(x,t) = t\cosh(x) \end{cases} \quad (3.16)$$

$$p^2 : \begin{cases} u_2(x,t) = \frac{t^2}{2!}\sinh(x) \\ v_2(x,t) = \frac{t^2}{2!}\cosh(x) \end{cases} \quad (3.17)$$

$$\vdots$$

and so on for other components. Using Eqs. (2.18)-(2.19), the series solutions are therefore given by

$$\begin{cases} u(x,t) = \sinh(x)\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \ldots\right) + \cosh(x)\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \ldots\right) \\ v(x,t) = \cosh(x)\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \ldots\right) + \sinh(x)\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \ldots\right) \end{cases} \quad (3.18)$$

using the Taylor expansion for sinht and coxht, we can find the exact solutions

$$\begin{cases} u(x,t) = \sinh(x+t) \\ v(x,t) = \cosh(x+t) \end{cases} \quad (3.19)$$

The inhomogeneous linear system: Consider the inhomogeneous linear system

$$u_t - v_x - (u - v) = -2 \quad (3.20)$$

$$v_t + u_x - (u - v) = -2 \quad (3.21)$$

with initial conditions

$$u(x,0) = 1 + e^x;\quad v(x,0) = -1 + e^x \quad (3.22)$$

Taking the Laplace transform on both sides of Eqs. (3.20)-(3.21), then by using the differentiation property of Laplace transform and initial conditions (3.22) gives

$$u(x,s) = \frac{1}{s} + \frac{e^x}{s} - \frac{2}{s^2} + \frac{1}{s}£\left[v_x + (u - v)\right] \quad (3.23)$$

$$v(x,s) = -\frac{1}{s} + \frac{e^x}{s} - \frac{2}{s^2} + \frac{1}{s}£\left[(u - v) - u_x\right] \quad (3.24)$$

Taking the inverse Laplace transform of both sides of the (3.23) and (3.24), we have

$$u(x,t) = 1 + e^{x} - 2t + £^{-1}\left[\frac{1}{s}£\left[v_x + (u - v)\right]\right] \quad (3.25)$$

$$v(x,t) = -1 + e^{x} - 2t + £^{-1}\left[\frac{1}{s}£\left[(u - v) - u_x\right]\right] \quad (3.26)$$

By using Homotopy Perturbation Transform Algorithm (HPTA) the solutions functions u(x,t) and v(x,t) is given by

$$u = \sum_{m=0}^{\infty} p^m u_m(x,t), \; v = \sum_{m=0}^{\infty} p^m v_m(x,t) \quad (3.27)$$

Invoking Eq. (3.27) in Eqs. (3.25)-(3.26), we have

$$\sum_{m=0}^{\infty} p^m u_m(x,t) = 1 + e^{x} - 2t + p\left(£^{-1}\left[\frac{1}{s}£\left[\left(\sum_{m=0}^{\infty} p^m v_m(x,t)\right)_x + \left(\sum_{m=0}^{\infty} p^m v_m(x,t) - \sum_{m=0}^{\infty} p^m v_m(x,t)\right)\right]\right]\right) \quad (3.28)$$

$$\sum_{m=0}^{\infty} p^m v_m(x,t) = -1 + e^{x} - 2t + p\left(£^{-1}\left[\frac{1}{s}£\left[\left(\sum_{m=0}^{\infty} p^m v_m(x,t) - \sum_{m=0}^{\infty} p^m v_m(x,t)\right) - \left(\sum_{m=0}^{\infty} p^m u_m(x,t)\right)_x\right]\right]\right) \quad (3.29)$$

On comparing the coefficients of like powers of p we get require solution components:

$$p^0 : \begin{cases} u_0(x,t) = 1 + e^{x} - 2t \\ v_0(x,t) = -1 + e^{x} - 2t \end{cases} \quad (3.30)$$

$$p^1 : \begin{cases} u_1(x,t) = te^{x} + 2t \\ v_1(x,t) = -te^{x} + 2t \end{cases} \quad (3.31)$$

$$p^2 : \begin{cases} u_2(x,t) = \frac{t^2}{2!}e^{x} \\ v_2(x,t) = \frac{t^2}{2!}e^{x} \end{cases} \quad (3.32)$$

and so on for other components. Using (2.18)-(2.19), the series solutions are therefore given by

$$\begin{cases} u(x,t) = 1 + e^{x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right) \\ v(x,t) = -1 + e^{x}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots\right) \end{cases} \quad (3.33)$$

that converges to the exact solutions

$$\begin{cases} u(x,t) = 1 + e^{x+t} \\ v(x,t) = -1 + e^{x-t} \end{cases} \quad (3.34)$$

The inhomogeneous nonlinear system: Consider the system of inhomogeneous nonlinear partial differential equations

$$u_t - u_x v - u = 1 \quad (3.35)$$

$$v_t + uv_x + v = 1 \quad (3.36)$$

with initial conditions

$$u(x,0) = e^{-x}, \quad v(x,0) = e^{x} \quad (3.37)$$

Taking the Laplace transform on both sides of Eqs. (3.35)-(3.36), then by using the differentiation property of Laplace transform and initial conditions (3.37) gives

$$u(x,s) = \frac{e^{-x}}{s} + \frac{1}{s^2} + \frac{1}{s}£\left[u_x v + u\right] \quad (3.38)$$

$$v(x,s) = \frac{e^{x}}{s} + \frac{1}{s^2} - \frac{1}{s}£\left[uv_x + v\right] \quad (3.39)$$

Applying inverse Laplace transform of both sides of the (3.38) and (3.39), we have

$$u(x,t) = e^{-x} + t + £^{-1}\left[\frac{1}{s}£\left[u_x v + \sum_{m=0}^{\infty} p^m u_m(x,t)\right]\right] \quad (3.40)$$

$$v(x,t) = e^{x} + t - £^{-1}\left[\frac{1}{s}£\left[uv_x + \sum_{m=0}^{\infty} p^m v_m(x,t)\right]\right] \quad (3.41)$$

We represent u(x,t) and v(x,t) by the infinite series (2.11) then inserting these series into both sides of Eqs. (3.38)-(3.39) yields

$$\sum_{m=0}^{\infty} p^m u_m(x,t) = e^{-x} + t + p\left(£^{-1}\left[\frac{1}{s}£\left[\sum_{m=0}^{\infty} p^m H_{1m}(u,v) + \sum_{m=0}^{\infty} p^m u_m(x,t)\right]\right]\right) \quad (3.42)$$

$$\sum_{m=0}^{\infty} p^m v_m(x,t) = e^{x} + t - p\left(£^{-1}\left[\frac{1}{s}£\left[\sum_{m=0}^{\infty} p^m H_{2m}(u,v) + \sum_{m=0}^{\infty} p^m v_m(x,t)\right]\right]\right) \quad (3.43)$$

where $H_{1m}(u,v)$ and $H_{2m}(u,v)$ are He's polynomials that represents nonlinear terms vu_x and uv_x respectively. We have a few terms of the He's polynomials for vu_x and uv_x, which are given by

$$\begin{aligned} H_{10}(u,v) &= v_0 u_{0x} \\ H_{11}(u,v) &= v_1 u_{0x} + v_0 u_{1x} \\ H_{12}(u,v) &= v_2 u_{0x} + v_1 u_{1x} + v_0 u_{2x} \\ &\vdots \end{aligned} \quad (3.44)$$

$$\begin{aligned} H_{20}(u,v) &= u_0 v_{0x} \\ H_{21}(u,v) &= u_1 v_{0x} + u_0 v_{1x} \\ H_{22}(u,v) &= u_2 v_{0x} + u_1 v_{1x} + u_0 v_{2x} \\ &\vdots \end{aligned} \quad (3.45)$$

Comparing the coefficients of like powers of p, we have

$$p^0 : \begin{cases} u_0(x,t) = e^{-x} + t \\ v_0(x,t) = e^{x} + t \end{cases} \quad (3.46)$$

$$p^1 : \begin{cases} u_1(x,t) = £^{-1}\left[\frac{1}{s}£[H_{10}(u,v) + u_0(x,t)]\right] \\ \quad = -t - \frac{t^2}{2!} + te^{-x} + \frac{t^2}{2!}e^{-x} \\ v_1(x,t) = -£^{-1}\left[\frac{1}{s}£[H_{20}(u,v) + v_0(x,t)]\right] \\ \quad = -t - \frac{t^2}{2!} - te^{x} + \frac{t^2}{2!}e^{x} \end{cases} \quad (3.47)$$

Proceeding in a similar manner, we have

$$p^2 : \begin{cases} u_2(x,t) = \frac{t^2}{2!} + t^2 e^{-x} + \ldots \\ v_2(x,t) = \frac{t^2}{2!} + t^2 e^{x} + \ldots \end{cases} \quad (3.48)$$

Similarly, we can find other components. The series solutions are therefore given by

$$\begin{cases} u(x,t) = e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right) \\ v(x,t) = e^{x}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots\right) \end{cases} \quad (3.49)$$

By using the Taylor expansion for e^t and e^{-t}, we can find the exact solutions of the above system of inhomogeneous nonlinear PDES as follows

$$\begin{cases} u(x,t) = e^{-x+t} \\ v(x,t) = e^{x-t} \end{cases} \quad (3.50)$$

CONCLUSION

In this work, a Homotopy perturbation transform algorithm which is based on the Homotopy perturbation is used to solve linear and nonlinear system of partial differential equations. The method presents a useful way to develop an analytic treatment for these system. The proposed scheme can be applied for system more than two linear and nonlinear partial differential equations with less computational work.

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