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Modified Decomposition Method for Systems of PDEs

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Abstract: In this paper, we apply modified decomposition method (MDM) to solve systems of partial differential equations. Several examples are given to verify the reliability and efficiency of the method. Numerical results clearly reveal the complete reliability of the proposed algorithm.

Key words: Modified decomposition method · Linear PDEs · Nonlinear problems · Systems of partial differential equations

INTRODUCTION

The partial differential equations are of great significance in the diversified physical problems related to physics, astrophysics, magnetic dynamics, water surface, gravity waves, ion acoustic waves in plasma, electromagnetic radiation reactions, engineering and applied sciences [1-16]. Several techniques including decomposition, homotopy perturbation, exp-function and variational iteration have been employed to solve such equations analytically and numerically, see [1-16] and the reference therein. Inspired and motivated by the ongoing research in this area, we applied modified decomposition method (MDM) which is mainly due to Geijji and Jafari [4] to solve system of differential equations. It is observed that the proposed technique is highly suitable for the solution of such problems. Several examples are given to verify the reliability, efficiency and accuracy of suggested MDM.

Modified Decomposition Method (MDM): Consider the following general functional equations:

$$f(x) = 0, \tag{1}$$

To convey the idea of the modified decomposition method [4], we rewrite the above equation as:

$$y = N(y)+c, \tag{2}$$

Where N is a nonlinear operator from a banach space $B \to B$ and f is a known function. We are looking for a solution of equation (1) having the series form:

$$y = \sum_{i=0}^{\infty} y_i. \tag{3}$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{i=0}^{\infty} y_i\right) = N(y_0) + \sum_{i=0}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\}. \tag{4}$$

From equations (3) and (4), equation (2) is equivalent to

$$\sum_{i=0}^{\infty} y_i = c + N(y_0) + \sum_{i=0}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\}. \tag{5}$$

We define the following recurrence relation:

$$\begin{cases} y_0 = c, \\ y_1 = N(y_0), \\ y_{m+1} = N(y_0 + \ldots + y_m) - N(y_0 + \ldots + y_{m-1}), \quad m = 1,2,3,\ldots, \end{cases} \tag{6}$$

then

$$(y_1 + \ldots + y_{m+1}) = N(y_0 + \ldots + y_m), \quad m = 1,2,3,\ldots,$$

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and
$$y = f + \sum_{i=1}^{\infty} y_i,$$
if N is a contraction, i.e.
$\| N(x) - N(y) \| \le \| x - y \|,\ 0 < K < 1$, then
$$\| y_{m+1} \| = \| N(y_0 + \ldots + y_m) - N(y_0 + \ldots + y_{m-1})$$
$$\| \le K \| y_m \| \le K^m \| y_0 \|,\ m = 0,1,2,3,\ldots,$$
and the series $\sum_{i=1}^{\infty} = y_i$ absolutely and uniformly converges to a solution of equation (1) [4, 11, 14-16], which is unique, in view of the Banach fixed-point theorem.

Numerical Applications: In this section, we apply the modified decomposition method (MDM) to solve systems of partial differential equations. Numerical results are very encouraging.

Example 3.1: Consider the following linear system of partial differential equations

$$u_t + v_x = 0,$$
$$v_t + u_x = 0,$$
With initial conditions

$$u(x,0) = e^x, \qquad v(x,0) = e^{-x}.$$

Applying the modified decomposition method (MDM), we get
$$u_{n+1}(x,t) = e^x - \int_0^t \frac{\partial v_n(x,\xi)}{\partial x} d\xi,$$
$$v_{n+1}(x,t) = e^{-x} - \int_0^t \frac{\partial u_n(x,\xi)}{\partial x} d\xi.$$

Consequently, following approximants are obtained

$$\begin{cases} u_0(x,t) = c, \\ u_0(x,t) = e^x, \\ v_0(x,t) = c, \\ v_0(x,t) = e^{-x}, \end{cases}$$

$$\begin{cases} u_1(x,t) = N u_0(x,t), \\ u_1(x,t) = e^x + te^{-x}, \\ v_1(x,t) = N u_0(x,t), \\ v_1(x,t) = e^{-x} - te^x, \end{cases}$$

$$\begin{cases} u_2(x,t) = N\big(u_0(x,t) + u_1(x,t)\big) - N u_0(x,t), \\ u_2(x,t) = e^x + te^{-x} + \frac{t^2}{2!} e^x, \\ v_2(x,t) = N\big(u_0(x,t) + u_1(x,t)\big) - N u_0(x,t), \\ v_2(x,t) = e^{-x} - te^x + \frac{t^2}{2!} e^{-x}, \end{cases}$$

$$\begin{cases} u_3(x,t) = N\big(u_0(x,t) + u_1(x,t) + u_2(x,t)\big) - N\big(u_0(x,t) + u_1(x,t)\big), \\ u_3(x,t) = e^x + te^{-x} + \frac{t^2}{2!} e^x + \frac{t^3}{3!} e^{-x}, \\ v_3(x,t) = N\big(u_0(x,t) + u_1(x,t) + u_2(x,t)\big) - N\big(u_0(x,t) + u_1(x,t)\big), \\ v_3(x,t) = e^{-x} - te^x + \frac{t^2}{2!} e^{-x} - \frac{t^3}{3!} e^x, \end{cases}$$

$\vdots$

The series solution is given as

$$u(x,t) = e^x \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots \right) + e^{-x} \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots \right),$$
$$v(x,t) = e^{-x} \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots \right) - e^x \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots \right),$$

and the closed form solution is given by

$$(u,v) = (e^x \cosh t + e^{-x} \sinh t,\ e^{-x} \cosh t - e^x \sinh t).$$

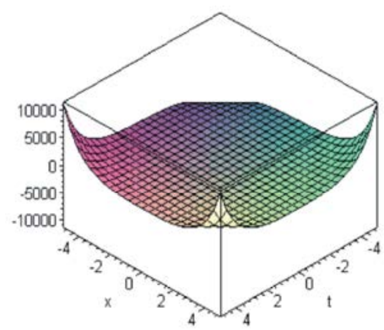

Fig. 1: (U)

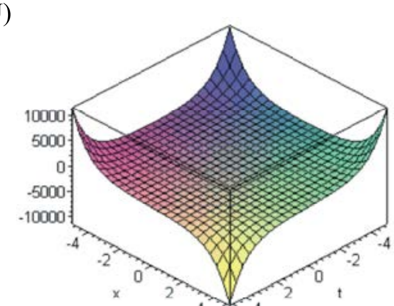

Fig. 2: (V)

Example 3.2: Consider the following linear system of partial differential equations

$$u_t + v_x - 2v = 0,$$
$$v_t + u_x + 2u = 0,$$

With initial conditions

$$u(x,0) = \sin x, \qquad v(x,0) = \cos x.$$

Applying the modified decomposition method (MDM), we get

$$u_{n+1}(x,t) = \sin x - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial x} - 2v_n \right) d\xi,$$

$$v_{n+1}(x,t) = \cos x - \int_0^t \left(\frac{\partial v_n(x,\xi)}{\partial x} + 2\tilde{u}_n \right) d\xi.$$

Consequently, following approximants are obtained

$$\begin{cases} u_0(x,t) = c, \\ u_0(x,t) = \sin x, \\ v_0(x,t) = c, \\ v_0(x,t) = \cos x, \end{cases}$$

$$\begin{cases} u_1(x,t) = N u_0(x,t), \\ u_1(x,t) = \sin x + t\cos x, \\ v_1(x,t) = N u_0(x,t), \\ v_1(x,t) = \cos x - t\sin x, \end{cases}$$

$$u_2(x,t) = N\big(u_0(x,t) + u_1(x,t)\big) - N u_0(x,t),$$
$$u_2(x,t) = \sin x + t\cos x - \frac{t^2}{2!}\sin x,$$
$$u_2(x,t) = N\big(u_0(x,t) + u_1(x,t)\big) - N u_0(x,t),$$
$$v_2(x,t) = \cos x - t\sin x - \frac{t^2}{2!}\cos x,$$

$$\begin{cases} u_3(x,t) = N\big(u_0(x,t) + u_1(x,t) + u_2(x,t)\big) - N\big(u_0(x,t) + u_1(x,t)\big), \\ u_3(x,t) = \sin x + t\cos x - \frac{t^2}{2!}\sin x - \frac{t^3}{3!}\cos x, \\ v_3(x,t) = N\big(u_0(x,t) + u_1(x,t) + u_2(x,t)\big) - N\big(u_0(x,t) + u_1(x,t)\big), \\ v_3(x,t) = \cos x - t\sin x - \frac{t^2}{2!}\cos x - \frac{t^3}{3!}\sin x, \end{cases}$$

$\vdots$

The series solution is given by

$$u(x,t) = \sin x\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots\right) + \cos x\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right),$$

$$v(x,t) = \cos x\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots\right) - \sin x\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right),$$

and the closed form solution is obtained as

$$(u,v) = (\sin(x+t), \cos(x+t)).$$

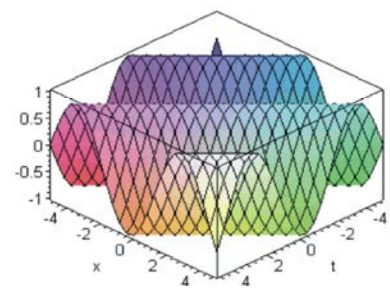

Fig. 3: (U)

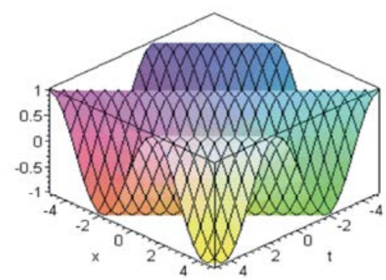

Fig. 4: (V)

Example 3.3 Consider the following nonlinear system of partial differential equations

$$u_t + v_x + u = 1,$$
$$v_t + u_x - v = 1,$$

With initial conditions

$$u(x,0) = e^x, \qquad v(x,0) = e^{-x}.$$

Applying the modified decomposition method (MDM), we get

$$u_{n+1}(x,t) = e^x - \int_0^t \left(v_n \left(u_n\right)_x + u_n - 1\right) d\xi,$$

$$v_{n+1}(x,t) = e^{-x} - \int_0^t \left(-u_n \left(v_n\right)_x - v_n - 1\right) d\xi,$$

Consequently, following approximants are obtained:

$$\begin{cases} u_0(x,t) = c, \\ u_0(x,t) = e^{x}, \\ v_0(x,t) = c, \\ v_0(x,t) = e^{-x}, \end{cases}$$

Proceeding as before, the series solution is given as

$$(u,v) = \left(e^{x}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right), e^{-x}\left(1 + t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right)\right),$$

and in the closed form solution is obtained as

$$(u,v) = (e^{x-t}, e^{-x+t}).$$

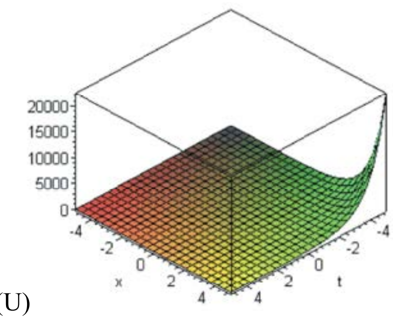

Fig. 5: (U)

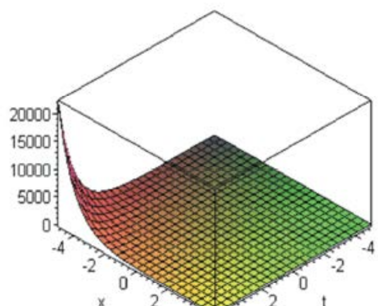

Fig. 6: (V)

Example 3.4: Consider the following nonlinear system of partial differential equations

$$\begin{aligned} u_t + v_x w_y - v_y w_x &= -u, \\ v_t + w_x u_y + w_y u_x &= v, \\ w_t + u_x v_y + u_y v_x &= -w, \end{aligned}$$

with initial conditions

$$u(x,y,0) = e^{x+y}, \quad v(x,y,0) = e^{x-y}, \quad w(x,y,0) = e^{-x+y}.$$

Applying the modified decomposition method (MDM), we get

$$\begin{cases} u_{n+1}(x,y,t) = e^{x+y} - \displaystyle\int_0^t \left(\left(\frac{\partial v_n}{\partial x}\right)\left(\frac{\partial w_n}{\partial y}\right) - \left(\frac{\partial v_n}{\partial y}\right)\left(\frac{\partial w_n}{\partial x}\right) - u_n \right) d\xi, \\ v_{n+1}(x,y,t) = e^{x-y} - \displaystyle\int_0^t \left(\left(\frac{\partial w_n}{\partial x}\right)\left(\frac{\partial u_n}{\partial y}\right) - \left(\frac{\partial w_n}{\partial y}\right)\left(\frac{\partial u_n}{\partial x}\right) - v_n \right) d\xi, \\ w_{n+1}(x,y,t) = e^{-x+y} - \displaystyle\int_0^t \left(\left(\frac{\partial u_n}{\partial x}\right)\left(\frac{\partial v_n}{\partial y}\right) - \left(\frac{\partial u_n}{\partial y}\right)\left(\frac{\partial v_n}{\partial x}\right) - w_n \right) d\xi. \end{cases}$$

Consequently, following approximants are obtained

$$\begin{cases} u_0(x,t) = c, \\ u_0(x,y,t) = e^{x+y}, \\ v_0(x,t) = c, \\ v_0(x,y,t) = e^{x-y}, \\ w_0(x,t) = c, \\ w_0(x,y,t) = e^{-x+y}, \end{cases}$$

$$\begin{cases} u_1(x,t) = N\,u_0(x,t), \\ u_1(x,y,t) = e^{x+y}(1-t), \\ v_1(x,t) = N\,u_0(x,t), \\ v_1(x,y,t) = e^{x-y}(1+t), \\ w_1(x,t) = N\,u_0(x,t), \\ w_1(x,y,t) = e^{-x+y}(1+t), \end{cases}$$

$$\begin{cases} u_2(x,t) = N\left(u_0(x,t) + u_1(x,t)\right) - N\,u_0(x,t), \\ u_2(x,y,t) = e^{x+y}(1 - t + \frac{t^2}{2!}), \\ v_2(x,t) = N\left(u_0(x,t) + u_1(x,t)\right) - N\,u_0(x,t), \\ v_2(x,y,t) = e^{x-y}(1 + t + \frac{t^2}{2!}), \\ w_2(x,t) = N\left(u_0(x,t) + u_1(x,t)\right) - N\,u_0(x,t), \\ w_2(x,y,t) = e^{-x+y}(1 + t + \frac{t^2}{2!}), \end{cases}$$

$$\begin{cases} u_3(x,t) = N\left(u_0(x,t) + u_1(x,t) + u_2(x,t)\right) - N\left(u_0(x,t) + u_1(x,t)\right), \\ u_3(x,y,t) = e^{x+y}(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!}), \\ u_3(x,t) = N\left(u_0(x,t) + u_1(x,t) + u_2(x,t)\right) - N\left(u_0(x,t) + u_1(x,t)\right), \\ v_3(x,y,t) = e^{x-y}(1 + t + \frac{t^2}{2!} + \frac{t3}{3!}), \\ u_3(x,t) = N\left(u_0(x,t) + u_1(x,t) + u_2(x,t)\right) - N\left(u_0(x,t) + u_1(x,t)\right), \\ w_3(x,y,t) = e^{-x+y}(1 + t + \frac{t^2}{2!} - \frac{t^3}{3!}), \end{cases}$$

$\vdots$

The closed form solution is given as

$$(u,v,w) = (e^{x+y-t}, e^{x-y+t}, e^{-x+y+t}).$$

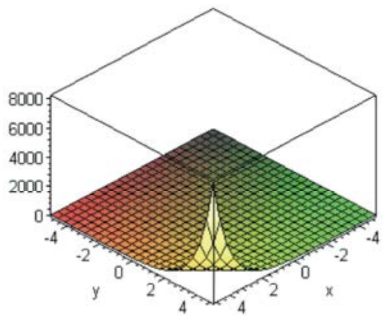

Fig. 7: (U, t = 1)

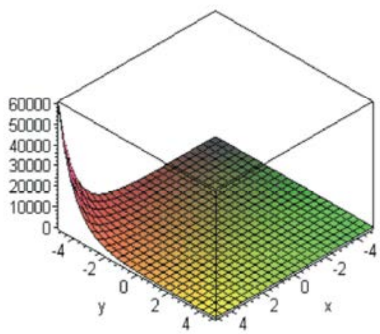

Fig. 8: (V, t = 1)

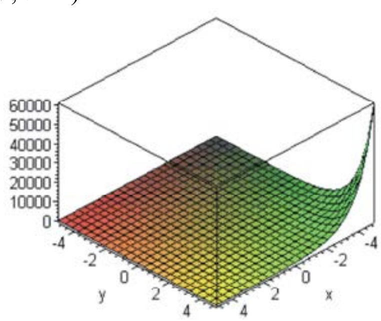

Fig. 9: (W, t = 1)

CONCLUSION

In this paper, we applied modified decomposition method (MDM) to solve systems of partial differential equations. The method is applied in a direct way without using linearization, transformation, discretization or restrictive assumptions. It may be concluded that the MDM is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. It is worth mentioning that the method is capable of reducing the volume of the computational work as compare to the classical methods while still maintaining the high accuracy of the numerical result.

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