

## Inverse Nodal Problem for Dirac Operator

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**Abstract:** Inverse nodal problems consist in constructing operators from the given zeros of their eigenfunctions. In this study, we have estimated nodal points and nodal lengths for Dirac operator. Furthermore, by using nodal points (zeros of eigenfunctions), we have shown that the potential functions of Dirac operator can be established uniquely.

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**Key words:** Dirac operator · Nodal Point · Reconstruction Formula

### INTRODUCTION

Inverse problems are studied for certain special classes of ordinary differential operators. Typically, in inverse eigenvalue problems, one measures the frequencies of a vibrating system and tries to infer some physical properties of the system. An early important result in this direction, which gave vital impetus for the further development of inverse problem theory, was obtained in [1-5]. In later years, inverse Sturm-Liouville problems were solved by using some new methods [6-9].

The Dirac equation is a modern presentation of the relativistic quantum mechanics of electrons intended to make new mathematical results accessible to a wider audience. It treats in some depth the relativistic invariance of a quantum theory, self-adjointness and spectral theory, qualitative features of relativistic bound and scattering states and the external field problem in quantum electrodynamics, without neglecting the interpretational difficulties and limitations of the theory.

Inverse problems for Dirac system had been investigated by Moses [10], Prats and Toll [11], Verde [12], Gasymov and Levitan [13] and Panakhov [14]. It is well known [15] that two spectra uniquely determine the matrix-valued potential function. In particular, in work [16], eigenfunction expansions for one dimensional Dirac operators describing the motion of a particle in quantum mechanics are investigated.

In some recent interesting works, J. R. McLaughlin [17], Hald and McLaughlin [18] and Browne and Sleeman [19] have taken a new approach to inverse spectral theory

for the Sturm-Liouville problem. The novelty of this work lies in the use of nodal points as the given spectral data. In later years, inverse nodal problems were studied by several authors [20-22].

In this work, we are concerned with the inverse problem for Dirac operator, using a new kind of spectral data, known as nodal points.

Let  $L$  denote a matrix operator

$$L = \begin{pmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{pmatrix}, \quad p_{12}(x) = p_{21}(x),$$

Where  $p_{ik}(x)$  ( $i, k = 1, 2$ ) are real functions which are defined and continuous on the interval  $[0, \pi]$ . Further, let  $\varphi(x, \lambda)$  denotes a two components vector function.

$$\varphi(x, \lambda) = \begin{pmatrix} \varphi_1(x, \lambda) \\ \varphi_2(x, \lambda) \end{pmatrix}.$$

Then the equation

$$\left( B \frac{d}{dx} + L - \lambda I \right) \varphi = 0,$$

Where  $\lambda$  is a parameter and

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

is equivalent to a system of two simultaneous first-order ordinary differential equations

$$\begin{aligned} \frac{d\varphi_2}{dx} + p_{11}(x)\varphi_1 + p_{12}(x)\varphi_2 &= \lambda\varphi_1, \\ -\frac{d\varphi_1}{dx} + p_{21}(x)\varphi_1 + p_{22}(x)\varphi_2 &= \lambda\varphi_2. \end{aligned} \quad (1.1)$$

For the case in which

$$p_{12}(x) = p_{21}(x) = 0,$$

$$p_{11}(x) = V(x) + m,$$

$$p_{22}(x) = V(x) - m$$

Where  $V(x)$  is a potential function and  $m$  is the mass of a particle, the system (1.1) is known in relativistic quantum theory as a stationary one-dimensional Dirac system. In case of  $p_{12}(x) = p_{21}(x) = 0$ , we obtain the following system called first canonic form of Dirac operator.

$$\begin{aligned} \varphi_2' - \{\lambda + p(x)\}\varphi_1 &= 0 \\ \varphi_1' - \{\lambda + r(x)\}\varphi_2 &= 0, \end{aligned} \quad (1.2)$$

With the following boundary conditions

$$\begin{aligned} \varphi_2(0, \lambda) \cos \alpha + \varphi_1(0, \lambda) \sin \alpha &= 0 \\ \varphi_2(\pi, \lambda) \cos \beta + \varphi_1(0, \lambda) \sin \beta &= 0, \end{aligned} \quad (1.3)$$

Where

$$\begin{aligned} \alpha &= \cos \left\{ \frac{1}{2} \int_0^x [p(\tau) + r(\tau)] d\tau \right\} \\ \beta &= \sin \left\{ \frac{1}{2} \int_0^x [p(\tau) + r(\tau)] d\tau \right\} \end{aligned}$$

Let us denote by  $\varphi(x, \lambda)$  the solution of the system (1.2) satisfying the initial conditions

$$\varphi_1(0, \lambda) = \cos \alpha, \quad \varphi_2(0, \lambda) = -\sin \alpha \quad (1.5)$$

The function  $\varphi(x, \lambda)$  obviously satisfies the following condition

$$\varphi_2(0, \lambda) \cos \alpha + \varphi_1(0, \lambda) \sin \alpha = 0. \quad (1.6)$$

We will assume that the functions  $p(x)$  and  $r(x)$  are continuous on the interval  $[0, \pi]$  and  $\lambda$  is an eigenvalue of this problem.

It is well known [13], for  $|\lambda| \rightarrow \infty$  the following estimates hold uniformly with respect to  $x, 0 \leq x \leq \pi$ .

$$\varphi_1(x, \lambda) = \cos \{ \xi(x, \lambda) - \alpha \} + O\left(\frac{1}{\lambda}\right)$$

$$\varphi_2(x, \lambda) = \sin \{ \xi(x, \lambda) - \alpha \} + O\left(\frac{1}{\lambda}\right)$$

Where

$$\xi(x, \lambda) = \lambda x - \frac{1}{2} \int_0^x [p(\tau) + r(\tau)] d\tau$$

and the sequence  $\{\lambda_n\}$  satisfies the classical asymptotic form [13]

$$\lambda_n = n + \frac{\vartheta}{\pi} + O\left(\frac{1}{n}\right), n = 0, \pm 1, \pm 2, \dots, \quad (1.7)$$

Where  $\vartheta$  is defined by

$$\vartheta = \beta - \alpha - \frac{1}{2} \int_0^\pi [p(\tau) + r(\tau)] d\tau.$$

Let  $\lambda_n, n \rightarrow \infty$  be the eigenvalues of Dirac operator and

$$0 < x_{1,1}^{(n)} < \dots < x_{j,1}^{(n)} < \pi, (0 < x_{1,2}^{(n)} < \dots < x_{j,2}^{(n)} < \pi)$$

$j = 1, 2, \dots, n - 1, n \in \mathbb{N}$  be the nodal points of the  $n$ th eigenfunctions.

**Main Results:** In this section, our purpose is to develop asymptotic expressions for the  $x_j^{(n)}$  and  $l_j^{(n)}, j = 1, 2, \dots, n$

$- 1, n \in \mathbb{N}$  at which  $\varphi_k, k = 1, 2$ , the eigenfunction corresponding to the eigenvalue  $\lambda_n$  of the problem (1.2)-(1.3) vanishes. Suppose that  $x_{j,1}^{(n)}, (x_{j,2}^{(n)})$  and  $l_{j,1}^{(n)}, (l_{j,2}^{(n)})$  are the nodal points and nodal lengths of the functions  $\varphi_1(x, \lambda)(\varphi_2(x, \lambda))$  respectively.

**Theorem 2.1:** We consider the equations

$$\begin{aligned} \varphi_2' - \{\lambda + p(x)\}\varphi_1 &= 0, \\ \varphi_1' - \{\lambda + r(x)\}\varphi_2 &= 0, \end{aligned} \quad (2.1)$$

With the following boundary conditions

$$\varphi_2(0, \lambda) \cos \alpha + \varphi_1(0, \lambda) \sin \alpha = 0$$

$$\varphi_2(\pi, \lambda) \cos \beta + \varphi_1(\pi, \lambda) \sin \beta = 0, \quad (2.2)$$

Where  $p(x)$  and  $r(x)$  are real and continuous functions on  $[0, \pi]$ . Then, the nodal points of the problem (2.1)-(2.2) are

$$x_j^{(n)} = \begin{pmatrix} x_{j,1}^{(n)} \\ x_{j,2}^{(n)} \end{pmatrix} = \begin{pmatrix} \frac{(j-\frac{1}{2})\pi}{\lambda_n} + \frac{1}{2\lambda_n} \int_0^{x_{j,1}^{(n)}} [p(\tau) + r(\tau)] d\tau + O\left(\frac{1}{\lambda_n}\right) \\ \frac{j\pi}{\lambda_n} + \frac{1}{2\lambda_n} \int_0^{x_{j,2}^{(n)}} [p(\tau) + r(\tau)] d\tau + O\left(\frac{1}{\lambda_n}\right) \end{pmatrix} \quad (2.3)$$

and the nodal lengths are

$$x_j^{(n)} = \begin{pmatrix} x_{j,1}^{(n)} \\ x_{j,2}^{(n)} \end{pmatrix} = \begin{pmatrix} \frac{\pi}{\lambda_n} + \frac{1}{2\lambda_n} \int_0^{x_{j,1}^{(n)}} [p(\tau) + r(\tau)] d\tau + O\left(\frac{1}{\lambda_n}\right) \\ \frac{\pi}{\lambda_n} + \frac{1}{2\lambda_n} \int_0^{x_{j,2}^{(n)}} [p(\tau) + r(\tau)] d\tau + O\left(\frac{1}{\lambda_n}\right) \end{pmatrix} \quad (2.4)$$

Where  $n \in \mathbb{N}$

**Proof:** The asymptotic expression for the eigenvalues [13] is

$$\lambda_n = n + \frac{\vartheta}{\pi} + O\left(\frac{1}{n}\right), n = 0, \pm 1, \pm 2, \dots, \quad (2.5)$$

Where  $\vartheta$  is defined by (1.8). Initially, we will develop asymptotic expressions of  $x_{j,1}^{(n)}$  and  $l_{j,1}^{(n)}$  for solution  $\varphi_1(x, \lambda)$ .

We use the classical estimate for  $|\lambda| \rightarrow \infty$

$$\varphi_1(x, \lambda) = \cos \{ \xi(x, \lambda) - \alpha \} + O\left(\frac{1}{\lambda}\right)$$

$$| \varphi_1(x, \lambda) - \cos \{ \xi(x, \lambda) - \alpha \} | \leq \frac{M}{\lambda},$$

Where  $M$  is a constant. Thence,  $\varphi_1(x, \lambda)$  will vanish in the intervals whose end points are solutions to.

$$\cos \{ \xi(x, \lambda) - \alpha \} = \pm \frac{M}{\lambda}.$$

This equation is equivalent to

$$\xi(x, \lambda) - \alpha = \arccos \left( \pm \frac{M}{\lambda} \right) = \left( j - \frac{1}{2} \right) \pi \pm \frac{M}{\lambda} + O\left(\frac{1}{\lambda^3}\right),$$

$$j = 1, 2, \dots, n-1$$

Where the last estimate has been obtained from the Taylor expansion for  $\arccos \left( \pm \frac{M}{\lambda} \right)$ . Thus, we get

$$\lambda x - \frac{1}{2} \int_0^x [p(\tau) + r(\tau)] d\tau - \alpha = \left( j - \frac{1}{2} \right) \pi \pm \frac{M}{\lambda} + O\left(\frac{1}{\lambda^3}\right)$$

and

$$x_{j,1}^{(n)} = \frac{\left( j - \frac{1}{2} \right) \pi}{\lambda_n} + \frac{1}{2\lambda_n} \int_0^{x_{j,1}^{(n)}} [p(\tau) + r(\tau)] d\tau + O\left(\frac{1}{\lambda_n}\right), \quad (2.6)$$

Where  $n = 1, 2, \dots$ ,

Hence, we obtain the nodal length for the function  $\varphi_1(x, \lambda)$  as

$$l_{j,1}^{(n)} = x_{j+1,1}^{(n)} - x_{j,1}^{(n)} = \frac{\pi}{\lambda_n} + \frac{1}{2\lambda_n} \int_{x_{j,1}^{(n)}}^{x_{j+1,1}^{(n)}} [p(\tau) + r(\tau)] d\tau + O\left(\frac{1}{\lambda_n}\right) \quad (2.7)$$

Now, we will develop asymptotic expressions of  $x_{j,2}^{(n)}$  and  $l_{j,2}^{(n)}$  for the function  $\varphi_2(x, \lambda)$ . Similarly, we use another classical estimate

$$\varphi_2(x, \lambda) = \sin \{ \xi(x, \lambda) - \alpha \} + O\left(\frac{1}{\lambda}\right)$$

$$| \varphi_2(x, \lambda) - \sin \{ \xi(x, \lambda) - \alpha \} | \leq \frac{N}{\lambda},$$

Where  $N$  is a constant. Thence,  $\varphi_2(x, \lambda)$  will vanish in the intervals whose end points are solutions to

$$\sin \{ \xi(x, \lambda) - \alpha \} = \pm \frac{N}{\lambda}$$

This equation is equivalent to

$$\xi(x, \lambda) - \alpha = \arcsin \left( \pm \frac{N}{\lambda} \right) = j\pi \pm \frac{N}{\lambda} + O\left(\frac{1}{\lambda^3}\right), j = 1, 2, \dots, n-1$$

Where this last estimate has been obtained from the Taylor expansion for  $\left( \pm \frac{N}{\lambda} \right)$ . Thus, we get

$$\lambda x - \frac{1}{2} \int_0^x [p(\tau) + r(\tau)] d\tau - \alpha = j\pi \pm \frac{N}{\lambda} + O\left(\frac{1}{\lambda^3}\right)$$

and

$$x_{j,2}^{(n)} = \frac{j\pi}{\lambda_n} + \frac{1}{2\lambda_n} \int_0^{x_{j,2}^{(n)}} [p(\tau) + r(\tau)] d\tau + O\left(\frac{1}{\lambda_n}\right), \quad (2.8)$$

Where  $j = 1, 2, \dots, n-1, n = 1, 2, \dots$

Hence, we obtain the nodal lengths for the function  $\varphi_2(x, \lambda)$  as

$$l_{j,2}^{(n)} = x_{j+1,2}^{(n)} - x_{j,2}^{(n)} = \frac{\pi}{\lambda_n} + \frac{1}{2\lambda_n} \int_{x_{j,2}^{(n)}}^{x_{j+1,2}^{(n)}} [p(\tau) + r(\tau)] d\tau + O\left(\frac{1}{\lambda_n}\right) \quad (2.9)$$

Therefore, we obtain the asymptotic expressions

$$x_j^{(n)} = \begin{pmatrix} x_{j,1}^{(n)} \\ x_{j,2}^{(n)} \end{pmatrix}, \quad l_j^{(n)} = \begin{pmatrix} l_{j,1}^{(n)} \\ l_{j,2}^{(n)} \end{pmatrix} \text{ at which } \varphi(x) = \begin{pmatrix} \varphi_1(x, \lambda) \\ \varphi_2(x, \lambda) \end{pmatrix}, \quad j = 1, 2, \dots,$$

$n-1, n = 1, 2, \dots$ , as following

$$x_j^{(n)} = \begin{pmatrix} \left( \frac{j-1}{2} \right) \pi + \frac{1}{2\lambda_n} \int_0^{x_{j,1}^{(n)}} [p(\tau) + r(\tau)] d\tau + O\left(\frac{1}{\lambda_n}\right) \\ \frac{j\pi}{\lambda_n} + \frac{1}{2\lambda_n} \int_0^{x_{j,2}^{(n)}} [p(\tau) + r(\tau)] d\tau + O\left(\frac{1}{\lambda_n}\right) \end{pmatrix}$$

and

$$l_j^{(n)} = \begin{pmatrix} \left( \frac{j-1}{2} \right) \pi + \frac{1}{2\lambda_n} \int_0^{x_{j,1}^{(n)}} [p(\tau) + r(\tau)] d\tau + O\left(\frac{1}{\lambda_n}\right) \\ \frac{j\pi}{\lambda_n} + \frac{1}{2\lambda_n} \int_0^{x_{j,2}^{(n)}} [p(\tau) + r(\tau)] d\tau + O\left(\frac{1}{\lambda_n}\right) \end{pmatrix}$$

This completes the proof.

**Uniqueness Theorems:** Now, we will give two uniqueness theorems for Dirac operator. We mentioned that this theorem was given for regular Sturm-Liouville problems by J.R. McLaughlin [17], Hald and McLaughlin [18], Browne and Sleeman [19].

**Theorem 3.1:** Assume that we have two Dirac problems of the type (2.1) with  $p, r, \alpha, \lambda$  and  $\tilde{p}, \tilde{r}, \tilde{\alpha}, \tilde{\lambda}$ . Let  $x_j^{(n)}$  and

$\tilde{x}_j^{(n)}$  be the nodes and  $x_j^{(n)} = \tilde{x}_j^{(n)}$  for a dense set of nodes. Then,  $\alpha$  is uniquely determined by any dense subset of the nodes on  $[0, \pi]$ .

**Proof:** Let consider the following equations,

$$\varphi_2' - \{\lambda + p(x)\}\varphi_1 = 0$$

$$\varphi_1' + \{\lambda + r(x)\}\varphi_2 = 0$$

and

$$\tilde{\varphi}_2' - \{\tilde{\lambda} + \tilde{p}(x)\}\tilde{\varphi}_1 = 0$$

$$\tilde{\varphi}_1' + \{\tilde{\lambda} + \tilde{r}(x)\}\tilde{\varphi}_2 = 0$$

With the initial conditions

$$\varphi_1(0, \lambda) = \cos \alpha, \varphi_2(0, \lambda) = -\sin \alpha$$

$$\tilde{\varphi}_1(0, \tilde{\lambda}) = \cos \tilde{\alpha}, \tilde{\varphi}_2(0, \tilde{\lambda}) = -\sin \tilde{\alpha}$$

Multiplying these equations by  $\tilde{\varphi}_1(x, \tilde{\lambda})$ ,  $-\tilde{\varphi}_2(x, \tilde{\lambda})$ ,  $-\varphi_1(x, \lambda)$  and  $\varphi_2(x, \lambda)$  respectively and adding, we obtain

$$\frac{d}{dx} \{\varphi_1(x, \lambda)\tilde{\varphi}_2(x, \tilde{\lambda}) - \varphi_2(x, \lambda)\tilde{\varphi}_1(x, \tilde{\lambda})\} = 0$$

Integrating this relation from 0 to  $x_j^{(n_k)}, k = \overline{1, 2}$  and using initial conditions, we yield that

$$\sin(\alpha - \tilde{\alpha}) = 0 \Rightarrow \alpha = \tilde{\alpha}$$

This completes the proof.

**Theorem 3.2:** Suppose that  $p$  and  $r$  are integrable functions on  $[0, \pi]$ . Then,  $\lambda, p - \int_0^\pi p$  and  $r - \int_0^\pi r$  are uniquely determined by any dense set of nodal points.

**Proof:** Assume that we have two problems of the type (2.1)-(2.2) with  $p, r, \alpha, \lambda$  and  $\tilde{p}, \tilde{r}, \tilde{\alpha}, \tilde{\lambda}$ . Let the nodal points  $x_j^{(n)}$

and  $\tilde{x}_j^{(n)}$  satisfying  $x_j^{(n)} = \tilde{x}_j^{(n)}$  ( $x_{j,1}^{(n)} = \tilde{x}_{j,1}^{(n)}$  and  $x_{j,2}^{(n)} = \tilde{x}_{j,2}^{(n)}$ )

form a dense set on  $[0, \pi]$ . We take solutions of (2.1)-(2.2)

as  $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$  for  $(p, r, \alpha, \lambda)$  and  $\tilde{\varphi} = \begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \end{pmatrix}$  for  $(\tilde{p}, \tilde{r}, \tilde{\alpha}, \tilde{\lambda})$ .

We'll show that  $\lambda = \tilde{\lambda}$ ,  $p = \tilde{p}$  and  $r = \tilde{r}$ . Let consider the following equations,

$$\varphi_2' - \{\lambda + p(x)\}\varphi_1 = 0 \quad (3.1)$$

$$\varphi_1' - \{\lambda + p(x)\}\varphi_2 = 0 \quad (3.2)$$

and

$$\tilde{\varphi}_2' - \{\tilde{\lambda} + \tilde{p}(x)\}\tilde{\varphi}_1 = 0 \quad (3.3)$$

$$\tilde{\varphi}_1' - \{\tilde{\lambda} + \tilde{r}(x)\}\tilde{\varphi}_2 = 0 \quad (3.4)$$

With boundary conditions

$$\varphi_1(0, \lambda) = \cos \alpha, \varphi_2(0, \lambda) = -\sin \alpha$$

$$\tilde{\varphi}_1(0, \tilde{\lambda}) = \cos \alpha, \tilde{\varphi}_2(0, \tilde{\lambda}) = -\sin \alpha$$

and

$$\varphi_1(\pi, \lambda) = \cos \beta, \varphi_2(\pi, \lambda) = -\sin \beta$$

$$\tilde{\varphi}_1(\pi, \tilde{\lambda}) = \cos \beta, \tilde{\varphi}_2(\pi, \tilde{\lambda}) = -\sin \beta$$

Firstly, let take into account the equations (3.1) and (3.3). Multiplying these equations  $\tilde{\varphi}_1(x, \tilde{\lambda})$ ,  $\varphi_1(\pi, \lambda)$  respectively and subtracting, we get

$$\begin{aligned} & \tilde{\varphi}_2'(x, \lambda)\tilde{\varphi}_1(x, \tilde{\lambda}) - \tilde{\varphi}_2'(x, \tilde{\lambda})\varphi_1(x, \lambda) = \\ & \{\lambda - \tilde{\lambda} + p - \tilde{p}\}\tilde{\varphi}_1(x, \tilde{\lambda})\varphi_1(x, \lambda) \end{aligned} \quad (3.5)$$

Recall that  $\tilde{\varphi}_1\varphi_1$  is uniformly bounded and  $x \in (0, \pi)$ . We choose a subsequence of nodes from the dense set. To show that  $\lambda = \tilde{\lambda}$ , we integrate both sides of (3.5) from  $x_{j,1}^{(n_k)}(x_{j,2}^{(n_k)})$  to  $\pi$  and choose a subsequence that tends to  $\pi$  to obtain

$$\frac{(\lambda - \tilde{\lambda}(\sin(2\beta)))}{2(\lambda + \tilde{\lambda})} = \int_{x_{j,1}^{(n_k)}}^x \{\lambda - \tilde{\lambda} + p - \tilde{p}\}\tilde{\varphi}_1(x, \tilde{\lambda})\varphi_1(x, \lambda)dx$$

From this results, we can conclude that  $\lambda = \tilde{\lambda}$ .

Let  $x_j^{(n_k)} = \tilde{x}_j^{(n_k)}$  and integrate both sides of (3.5) from 0 to  $x_{j,1}^{(n_k)}(x_{j,2}^{(n_k)})$ ,  $k = \overline{1, 2}$ ,  $n \in \mathbb{N}$ , we find

$$\int_0^{x_{j,1}^{(n_k)}} (\tilde{\varphi}_2'\tilde{\varphi}_1' - \tilde{\varphi}_2'\varphi_1')dx = \int_0^{x_{j,1}^{(n_k)}} \{\lambda - \tilde{\lambda} + p - \tilde{p}\}\tilde{\varphi}_1(x, \tilde{\lambda})\varphi_1(x, \lambda)dx$$

and

$$0 = \int_0^{x_{j,1}^{(n_k)}} \{p - \tilde{p}\}\tilde{\varphi}_1(x, \tilde{\lambda})\varphi_1(x, \lambda)dx$$

We take a sequence  $x_{j,1}^{(n_k)} = (x_{j,2}^{(n_k)})$ , accumulating at an arbitrary  $x \in (0, \pi)$ . Hence

$$0 = \int_0^x \left( p - \tilde{p} - \int_0^x (\tilde{p} - p)ds \right) \tilde{\varphi}_1(x, \tilde{\lambda})\varphi_1(x, \lambda)dx$$

and this holds for all  $x$ . We can therefore conclude  $p - \int_0^\pi p(s)ds$  is uniquely determined by a dense set of nodes by using Riemann-Lebesgue theorem.

Analogously, if we use the equations (3.2)-(3.4) and by above procees, we get

$$0 = \int_0^{x_{j,q}^{(n_k)}} \{r - \tilde{r}\}\tilde{\varphi}_2(x, \tilde{\lambda})\varphi_2(x, \lambda)dx$$

We take a sequence  $x_{j,q}^{(n_k)}$  ( $q = 1, 2$ ) accumulating at an arbitrary  $x \in (0, \pi)$ . Hence

$$0 = \int_0^x \left( r - \tilde{r} - \int_0^x (\tilde{r} - r)ds \right) \tilde{\varphi}_2(x, \tilde{\lambda})\varphi_2(x, \lambda)dx$$

holds for all  $x$ . By using Riemann-Lebesgue theorem, we conclude that  $r - \int_0^\pi r(s)ds$  is uniquely determined by a dense set of nodes.

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