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The Nehari Manifold for a Class of Elliptic Equations of P-laplacian Type

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Abstract: In this paper we prove the existence of positive solutions for a class of quasilinear elliptic equations of the form:

$$-\Delta_p u(u) = \lambda h(x)|u|^{r+1} dx + g(x)|u|^{s+1} dx, \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega,$$

in $W_0^{1,p}(\Omega)$ where λ is a real parameter, Ω is a bounded domain with smooth boundary in R^N, $N \geq 3$. and$1< r < p - 1 < s < (Np - N + p) / (N - P)$.

Key words: Nehari manifold · Minimizing sequence · Critical Sobolev exponent

INTRODUCTION

In this paper, we prove the existence of at least two positive solutions of the following Dirichlet elliptic problem:
$$\begin{cases} -\Delta_p u(x) = \lambda h(x)|u|^{r+1} dx + g(x)|u|^{s+1} dx & in\,\Omega, \\ u = 0 & on\,\partial\Omega. \end{cases} \qquad (E_\lambda)$$

Where Ω is a bounded domain in R^N. In order to state our main theorem, let us introduce the structure of problem (E_λ).

Assume that $N \geq 3$ and $2 \leq p < N$ and. $1 < r < p-1 < s < (Np - N + p)/(N - p)$ Let Ω be a bounded domain in R^N having C^2 boundary $\delta\Omega$.

Problem (E_λ) had been studied by Afrouzi and Khademloo in [1] in the case when p = 2.

Regarding the functions h and g, we assume that

(H) $h(\times) \geq 0$ for all $x \in \Omega$ and

$$h \in L^{r_0}(\Omega) \cap L^\infty(\Omega) \cap C^0(\Omega)$$

Where $\frac{1}{r_0} + \frac{r+1}{p^*} = 1,$ that is $r_0 = \frac{Np}{Np - (r+1)(N-p)}.$

(G) $g(\times) < 0$ a.e $x \in \Omega$ and $g \in L^{s_0}(\Omega) \cap L^\infty(\Omega)$

Where $\frac{1}{s_0} + \frac{s+1}{p^*} = 1,$ that is

$$s_0 = \frac{Np}{Np - (s+1)(N-p)}, \left(p^* = \frac{Np}{N-p}\right).$$

Next, we define $X = W_0^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the Norm

$$\|u\|_X = \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}.$$

We consider the energy functional $J_\lambda(\mu)$ for each $u \in X$

$$J_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{s+1}\int_\Omega g(x)|u|^{s+1} dx - \frac{\lambda}{r+1}\int_\Omega h(x)|u|^{r+1} dx.$$

It is well known that the solutions of Eq. (E_λ) are the critical points of the energy functional $J_\lambda(\mu)$.

Let S be the best Sobolev constant for the embedding of $W_0^{1,p}(\Omega)$ in $L^{p^*}(\Omega)$. We prove that Eq. (E_λ) has at least two positive solutions for λ in a suitable range. Our main result is:

Theorem 1.1: There exists $\lambda_0 > 0$ such that for $\lambda \in (0,\lambda_0)$, Eq. ($E_\lambda$) has at least two positive solutions.

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Notations and Preliminaries: First, we consider the Nehari minimization problem:

$$\alpha_\lambda = \inf\left\{ J_\lambda(u) \,\middle|\, u \in M_\lambda \right\},$$

Where $M_\lambda = \left\{ u \in X \setminus \{0\} \,\middle|\, \left\langle J'_\lambda(u), u \right\rangle = 0 \right\}, \quad \lambda > 0.$

Define

$$\psi_\lambda(u) = \left\langle J'_\lambda(u), u \right\rangle = \|u\|_X^p - \int_\Omega g(x)|u|^{s+1}\,dx - \lambda \int_\Omega h(x)|u|^{r+1}\,dx.$$

Then for $u \in M_\lambda$

$$\left\langle \psi'_\lambda(u), u \right\rangle = p\|u\|_X^p - (s+1) \int_\Omega g(x)|u|^{s+1}\,dx - \lambda(r+1)\int_\Omega h(x)|u|^{r+1}\,dx.$$

Similarly to the method used in [2], we split M_λ into three parts:

$$M_\lambda^+ = \left\{ u \in M_\lambda \,\middle|\, \left\langle \psi'_\lambda(u), u \right\rangle > 0 \right\},$$
$$M_\lambda^- = \left\{ u \in M_\lambda \,\middle|\, \left\langle \psi'_\lambda(u), u \right\rangle = 0 \right\},$$
$$M_\lambda^0 = \left\{ u \in M_\lambda \,\middle|\, \left\langle \psi'_\lambda(u), u \right\rangle < 0 \right\}.$$

Then, We Have the Following Results

Lemma 2.1: There exists $\lambda_1 > 0$ such that for each $\lambda \in (0, \lambda_1)$, we have $M_\lambda^0 = 0.$

Proof: We consider the following two cases:

case (I). $u \in M_\lambda$ and $\int_\Omega h(x)|u|^{r+1}\,dx = 0.$

Then

$$\|u\|_X^p - \int_\Omega g(x)|u|^{s+1}\,dx = 0.$$

Thus we have

$$\left\langle \psi_\lambda(u), u \right\rangle = p\|u\|_X^p - (s+1) \int_\Omega g(x)|u|^{s+1}\,dx = (p-s-1)\|u\|_X^p < 0,$$

and so $u \notin M_\lambda^0$

case (II). $u \in M_\lambda$ and $\int_\Omega h(x)|u|^{r+1}\,dx \neq 0.$ Suppose that $M_\lambda^0 \neq \varnothing$ for all $\lambda > 0$

For $u \in M_\lambda^0$, we have

$$0 = \left\langle \psi'_\lambda(u), u \right\rangle = p\|u\|_X^p - (s+1)\int_\Omega g(x)|u|^{s+1}\,dx - \lambda(r+1)\int_\Omega h(x)|u|^{r+1}\,dx = (p-r-1)\|u\|_X^p - (s-r)\int_\Omega g(x)|u|^{s+1}\,dx.$$

Thus,

$$\|u\|_X^p = \frac{s-r}{p-r-1}\int_\Omega g(x)|u|^{s+1}\,dx, \tag{1}$$

and

$$\lambda\int_\Omega h(x)|u|^{r+1}\,dx = \|u\|_X^p - \int_\Omega g(x)|u|^{s+1}\,dx = \frac{s+1-p}{p-r-1}\int_\Omega g(x)|u|^{s+1}\,dx. \tag{2}$$

Moreover

$$\left(\frac{s+1-p}{s-r}\right)\|u\|_X^p = \|u\|_X^p - \int_\Omega g(x)|u|^{s+1}\,dx = \lambda\int_\Omega h(x)|u|^{r+1}\,dx \le \lambda\|h\|_{L^{r_0}}\|u\|_{L^{p^*}(\Omega)}^{r+1} \le \lambda\|h\|_{L^{r_0}(\Omega)}\frac{1}{S^{\frac{r+1}{p}}}\|u\|_X^{r+1}.$$

This implies that

$$\|u\|_X \le \left[\lambda\left(\frac{s-r}{s+1-p}\right)\|h\|_{L^{r_0}}\frac{1}{S^{\frac{r+1}{p}}}\right]^{\frac{1}{p-r-1}}. \tag{3}$$

Let $I_\lambda : M_\lambda \to R$ be given by

$$I_\lambda(u) = k(p,r,s)\left[\frac{\|u\|_X^{p(s-p+2)}}{\int_\Omega g(x)|u|^{s+1}\,dx}\right]^{\frac{1}{s+1-p}} - \left(\lambda\int_\Omega h(x)|u|^{r+1}\,dx\right),$$

Where $k(p,r,s) = \left(\frac{p-r-1}{s-r}\right)^{\frac{s-p+2}{s+1-p}}\left(\frac{s+1-p}{p-r-1}\right).$ Then from (1) and (2) it follows that

$$I_\lambda(u) = 0 \tag{4}$$

for all $u \in M_\lambda^0$.

However, by (3), the Holder and Sobolev inequalities we drive

$$I_\lambda(u) \ge k(p,r,s)\left[\frac{\|u\|_X^{p(s-p+2)}}{\int_\Omega g(x)|u|^{s+1}\,dx}\right]^{\frac{1}{s+1-p}} - \lambda\|h\|_{L^{r_0}}\frac{1}{S^{\frac{r+1}{p}}}\|u\|_X^{r+1}$$

$$\ge k(p,r,s)\left[\frac{\|u\|_X^{p(s-p+2)}}{\|g\|_{L^{s_0}}\frac{1}{S^{\frac{s+1}{p}}}\|u\|_X^{s+1}}\right]^{\frac{1}{s+1-p}} -\lambda\|h\|_{L^{r_0}}\frac{1}{S^{\frac{r+1}{p}}}\|u\|_X^{r+1},$$

for $u\in M_\lambda^0$. Since

$$\frac{p(s-p+2)}{s-p+1} > p > r+1$$

there exists a constant C > 0 such that

$$\|u\|_X^{r+1} \le C\|u\|_X^{\frac{p(s-p+2)}{s+1-p}}.$$

Therefore

$$I_\lambda(u) \ge \|u\|_X^{r+1}\left[k(p,r,s)\left[\frac{S^{\frac{r+1}{p}}}{\|g\|_{L^{s_0}}}\right]^{\frac{1}{s+1-p}} C\|u\|_X^{\frac{-(s+1)}{s-p+1}} -\lambda\|h\|_{L^{r_0}}\frac{1}{S^{\frac{r+1}{p}}}\right]$$

$$\ge \|u\|_X^{r+1}\left\{k(p,r,s)\left[\frac{S^{\frac{r+1}{p}}}{\|g\|_{L^{s_0}}}\right]^{\frac{1}{s+1-p}} C\lambda^{\frac{-(s+1)}{(s-p+1)(p-r-1)}}\right.$$

$$\left.\left[\left(\frac{s-r}{s-p+1}\right)\|h\|_{L^{r_0}}\frac{1}{S^{\frac{r+1}{p}}}\right]^{\frac{-(s-1)}{(s-p+1)(p-r-1)}} -\lambda\|h\|_{L^{r_0}}\frac{1}{S^{\frac{r+1}{p}}}\right\}.$$

This implies that for λ sufficiently small we have $I_\lambda(u) > 0$ for all $u\in M_\lambda^0$, which contradicts (4).

Thus, we can conclude that there exists $\lambda_1 > 0$ such that for $\lambda \in (0, \lambda_1)$ we have $u\in M_\lambda^0$

By Lemma (2.1), for $\lambda \in (0, \lambda_1)$ we shall write $M_\lambda = M_\lambda^+ \cup M_\lambda^-$

and define $\alpha_\lambda^+ = \inf_{u\in M_\lambda^+} J_\lambda(u)$, $\alpha_\lambda^- = \inf_{u\in M_\lambda^-} J_\lambda(u)$.

The following Lemma shows that the minimizer on $M_\lambda(\Omega)$ are usually critical points of J_λ.

Lemma 2.2: For each $\lambda \in (0, \lambda_1)$ if u_0 is a local minimizer for J_λ on M_λ then $J'_\lambda(u) = 0$ in X^{-1}.

Proof: If u_0 is a local minimizer for J_λ on M_λ then u_0 is a solution of the optimization problem

Minimize $J_\lambda(u)$ subject to $\Psi_\lambda(u) = 0$

Hence, by the theory of Lagrange multipliers, there exists $\Lambda \in R$ such that

$$J'_\lambda(u_0) = \Lambda\psi'_\lambda(u_0) \quad \text{in} \quad X^{-1}.$$

Thus

$$\langle J'_\lambda(u_0), u_0\rangle = \Lambda\langle\psi'_\lambda(u_0), u_0\rangle = 0. \tag{5}$$

But $\langle\psi'_\lambda(u_0), u_0\rangle \ne 0$, since $u_0 \notin M_\lambda^0$.

Thus $\Lambda = 0$ which completes the proof.

Lemma 2.3: If $u\in M_\lambda^+$, then $\int_\Omega h(x)|u|^{r+1}dx > 0.$

Proof: We have

$$\|u\|_X^p - \int_\Omega g(x)|u|^{s+1}dx - \lambda\int_\Omega h(x)|u|^{r+1}dx = 0,$$

and

$$\|u\|_X^p > \frac{s-r}{p-r-1}\int_\Omega g(x)|u|^{s+1}dx.$$

Thus

$$\lambda\int_\Omega h(x)|u|^{r+1}dx = \|u\|_X^p - \int_\Omega g(x)|u|^{s+1}dx$$

$$> \frac{s+1-p}{p-r-1}\int_\Omega g(x)|u|^{s+1}dx > 0,$$

Which completes the proof.

For each $u \in X\setminus\{0\}$ we have

$$t_{\max} = \left[\frac{(p-r-1)\|u\|_X^p}{(s-r)\int_\Omega g(x)|u|^{s+1}dx}\right]^{\frac{1}{s+1-p}} > 0,$$

then the following Lemma holds:

Lemma 2.4: Let

$$\lambda_2 = \left(\frac{p-r-1}{s-r}\right)^{\frac{p-r-1}{s+1-p}} S^{\frac{p(s-r)}{s+1-p}}\|h\|_{L^{r_0}}^{-1},$$

then for $u \in X\setminus\{0\}$ and $\lambda \in (0, \lambda_1)$ we have

- If $\lambda\int_\Omega h(x)|u|^{r+1}dx \le 0,$ there exists a unique $t^-u \in M_\lambda^-$ and

$$J_\lambda(t^-u) = \sup_{t>t_{\max}} J_\lambda(tu).$$

- If $\lambda\int_\Omega h(x)|u|^{r+1}dx > 0,$ there exists a unique $0 < t^+ = t^+(u) < t_{\max}$, such that $t^+u \in M_\lambda^+$ and

$$J_\lambda\left(t^+u\right) = \inf_{0\le t\le t^-} J_\lambda(tu).$$

Proof: Fix $u \in X\setminus\{0\}$, let

$$s(t) = t^{p-r-1}\|u\|_X^p - t^{s-r}\int_\Omega g(x)|u|^{s+1}dx \quad \forall t \ge 0.$$

Then $s(0) = 0$, $s(t) \to -\infty$ as $s(t)$ is concave and achieves its maximum at $t_{\max}$ Moreover

$$s(t_{\max}) = \left(\frac{(p-r-1)\|u\|_X^p}{(s-r)\int_\Omega g(x)|u|^{s+1}dx}\right)^{\frac{p-r-1}{s+1-p}}\|u\|_X^p$$

or

$$-\left[\frac{(p-r-1)\|u\|_X^p}{(s-r)\int_\Omega g(x)|u|^{s+1}dx}\right]^{\frac{p-r-1}{s+1-p}}\int_\Omega g(x)|u|^{s+1}dx$$

$$=\|u\|_X^{r+1}\left[\left(\frac{(p-r-1)\|u\|_X^{s+1}}{(s-r)\int_\Omega g(x)|u|^{s+1}dx}\right)^{\frac{p-r-1}{s+1-p}}\right.$$

$$\left.-\left(\frac{(p-r-1)\|u\|_X^{\frac{(s+1)(p-r-1)}{s-r}}}{(s-r)\left(\int_\Omega g(x)|u|^{s+1}dx\right)^{\frac{p-r-1}{s-r}}}\right)^{\frac{s-r}{s+1-p}}\right]$$

$$=\|u\|_X^{r+1}\left[\left(\frac{p-r-1}{s-r}\right)^{\frac{p-r-1}{s+1-p}} - \left(\frac{p-r-1}{s-r}\right)^{\frac{s-r}{s+1-p}}\right]$$

$$\left[\frac{\|u\|_X^{s+1}}{\int_\Omega g(x)|u|^{s+1}dx}\right]^{\frac{p-r-1}{s+1-p}}$$

$$\ge \|u\|_X^{r+1}\left(\frac{p-r-1}{s-r}\right)^{\frac{p-r-1}{s+1-p}}\left[1-\left(\frac{p-r-1}{s-r}\right)\right]\left(\frac{S^{\frac{s+1}{p}}}{\|g\|_{L^{s_0}}}\right)^{\frac{p-r-1}{s+1-p}},$$

$$s(t_{\max}) \ge$$

$$\|u\|_X^{r+1}\left(\frac{p-r-1}{s-r}\right)^{\frac{p-r-1}{s+1-p}}\left[\left(\frac{s+1-p}{s-r}\right)\right]\left(\frac{S^{\frac{s+1}{p}}}{\|g\|_{L^{s_0}}}\right)^{\frac{p-r-1}{s+1-p}} \quad (6)$$

So

- If $\int_\Omega h(x)|u|^{r+1}dx \le 0,$ there exists a unique $t^- > t_{\max}$ such that $s(t^-) = \int_\Omega h(x)|u|^{r+1}dx$ and $s'(t^-) < 0.$

  Now,

$$(p-r-1)\left\|t^-u\right\|_X^p - (s-r)\int_\Omega g(x)\left|t^-u\right|^{s+1}dx$$

$$=\left(t^-\right)^{r+2}\left[(p-r-1)\left(t^-\right)^{p-r-2}\|u\|_X^p\right.$$

$$\left.-(s-r)\left(t^-\right)^{s-r-1}\int_\Omega g(x)|u|^{s+1}dx\right]$$

$$=\left(t^-\right)^{r+2}s'\left(t^-\right) < 0,$$

and

$$\left\langle J_\lambda'\left(t^-u\right), t^-u\right\rangle = \left(t^-\right)^p\|u\|_X^p - \left(t^-\right)^{s+1}\int_\Omega g(x)|u|^{s+1}dx$$

$$-\left(t^-\right)^{r+1}\lambda\int_\Omega h(x)|u|^{r+1}dx = \left(t^-\right)^{r+1}\left[s\left(t^-\right) - \lambda\int_\Omega h(x)|u|^{r+1}dx\right] = 0.$$

Thus $t^-u \in M_\lambda^-.$ On the other hand for $t > t_{max}$ we have

$$(p-r-1)\|tu\|_X^p - (s-r)\int_\Omega g(x)|tu|^{s+1}dx < 0,$$

$$\frac{d^2}{dt^2}J_\lambda(tu) < 0,$$

$$\frac{d}{dt}J_\lambda(tu) = t\|u\|_X^p - t^s\int_\Omega g(x)|u|^{s+1}dx - t^r\lambda\int_\Omega h(x)|u|^{r+1}dx,$$

thus, $J_\lambda\left(t^-u\right) = \sup_{t\ge t_{\max}} J_\lambda(tu).$

- If $\int_\Omega h(x)|u|^{r+1}dx > 0,$ by (6) we have

$$s(0) = 0 < \lambda \int_{\Omega} h(x)|u|^{r+1} dx$$
$$\leq \lambda \|h\|_{L^{r_0}} \frac{1}{S^{\frac{r+1}{p}}} \|u\|_X^{r+1}$$
$$< \|u\|_X^{r+1} \left(\frac{p-r-1}{s-r}\right)^{\frac{p-r-1}{s+1-p}} \left[\left(\frac{s+1-p}{s-r}\right)\right] \left(\frac{S^{\frac{s-1}{p}}}{\|g\|_{L^{s_0}}}\right)^{\frac{p-r-1}{s+1-p}}$$
$$\leq s(t_{\max}) \quad for \quad \lambda \in (0, \lambda_2)$$

So there exist unique t^+ and t^- such that $0 < t^+$

$$0 < t^+ < t_{\max} < t^-$$

and
$$s(t^+) = \lambda \int_{\Omega} h(x)|u|^{r+1} dx = s(t^-),$$
$$s'(t^+) > 0 > s'(t^-).$$

Thus $t^+u \in M_\lambda^+, \; t^-u \in M_\lambda^-,$

$J_\lambda(t^-u) \geq J_\lambda(tu) \geq J_\lambda(t^+u)$ for each $t \in [t^+, t^-]$ and $J_\lambda(t^+u) \leq J_\lambda(tu)$ for each $t \in [0, t^+]$. Thus

$$J_\lambda(t^-u) = \sup_{t \geq t_{\max}} J_\lambda(tu),$$
$$J_\lambda(t^+u) = \inf_{0 \leq t \leq t^-} J_\lambda(tu),$$

This completes the proof.

Let $\theta = \{x \in \Omega | \, h(x) > 0\}$. We know that θ is a open set in R^N, because that $h \in C^\circ(\Omega)$ Consider the following elliptic equation

$$\begin{cases} -\Delta_p u = g(x)|u|^{s-1}u, & in\theta, \\ 0 \leq u, & in\theta, \\ u = 0, & on\partial\theta, \end{cases} \tag{E_θ}$$

$$k(u) = \frac{1}{p}\int_\theta |\nabla u|^p dx - \frac{1}{s+1}\int_\theta g(x)|u|^{s+1} dx,$$

and the Nehari minimization problem

$$\beta(\theta) = \inf\{k(u) | \, u \in N\},$$

where $N = \{u \in X(\theta) \setminus \{0\} | \langle k'(u), u \rangle = 0\}$ and $X(\theta) = W_0^{1,p}(\theta)$. Then we have the following results.

Lemma 2.5: Equation (E_θ)has a positive solution w_0 uch that $k(w_0) = \beta(\theta) > 0$

Proof: First, we need to show that k is bounded below on N and $\beta > 0$ For $u \in N$

$$\int_\theta |\nabla u|^p dx = \int_\theta g(x)|u|^{s+1} dx$$
$$\leq \|g\|_{L^{s_0}} \frac{1}{S^{\frac{s+1}{p}}} \left(\int_\theta |\nabla u|^p dx\right)^{\frac{s+1}{p}}.$$

This implies

$$\int_\theta |\nabla u|^p dx \geq \left(\frac{S^{\frac{s+1}{p}}}{\|g\|_{L^{s_0}}}\right)^{\frac{p}{s+1-p}}. \tag{7}$$

Hence for all $u \in N$

$$k(u) = \frac{1}{p}\int_\theta |\nabla u|^p dx - \frac{1}{s+1}\int_\theta g(x)|u|^{s+1} dx$$
$$\geq \left(\frac{1}{p} - \frac{1}{s+1}\right)\left(\frac{S^{\frac{s+1}{p}}}{\|g\|_{L^{s_0}}}\right)^{\frac{p}{s+1-p}} > 0.$$

This implies $\beta > 0$ Let $(w_n\}$ be a minimizing sequence for k on N, then by (7) and the compact embedding theorem, without loss of generality we may assume that there exist w_0 in $X(\Theta)$ such that $w_n \xrightarrow{w} w_0$ weakly in $X(\Theta)$ and $w_n \longrightarrow w_0$ strongly in L^{s+1} (8)

We claim that $\int_\theta g(x)|w_0|^{s+1} dx > 0$. Otherwise by (8) we can conclude that

$$\int_\theta g(x)|w_n|^{s+1} dx \to 0 \qquad \text{as} \qquad n\to\infty$$

Thus $\|w_n\|_{X(\theta)}^p = 0(1)$ and

$$k(w_n) = \frac{1}{p}\int_\theta |\nabla w_n|^p dx - \frac{1}{s+1}\int_\theta g(x)|w_n|^{s+1} dx \to 0$$

as $n\to\infty$

This contradicts $k(w_n) \to \beta > 0$ as $n\to\infty$ Thus, $\int_\theta g(x)|w_0|^{s+1}dx > 0.$ In particular, $w_0 \neq 0$ Now, we shall prove that $w_n \to w_0$ strongly in $X(\theta)$ therwise we have

$$\int_\theta |\nabla w_0|^p dx < \liminf_{n\to\infty} \int_\theta |\nabla w_n|^p dx$$

and hence

$$\int_\theta |\nabla w_0|^p dx - \int_\theta g(x)|w_0|^{s+1} dx$$
$$< \liminf_{n\to\infty}\left(\int_\theta |\nabla w_n|^p dx - \int_\theta g(x)|w_n|^{s+1} dx\right) = 0.$$

By $\int_\theta g(x)|w_0|^{s+1}dx,$ there exist a unique $t_0 \neq 1$such that $t_0w_0 \in N$ Thus, $t_0 w_n \xrightarrow{w} t_0 w_0$ weakly in $X(\theta)$ and so

$$k(t_0w_0) < k(w_0) < \lim_{n\to\infty} k(w_n) = \beta,$$

Which is a contradiction. Hence $w_n \to w_0$ strongly in $X(\theta)$. This implies $w_0 \in N$ and

$$k(w_n \to k(w_0) = \beta \text{ as } n \to \infty$$

We may assume that w_0 is a positive solution of Eq. (E_θ).

Lemma 2.6:

- There exists $t_\lambda > 0$ such that

$$\alpha_\lambda \le \alpha_\lambda^+ < -\frac{p-r-1}{r+1} t_\lambda^2 \beta(\theta) < 0,$$

- J_λ is coercive and bounded bellow on M_λ for all $\lambda \in \left(0, \frac{s+1-p}{s-r}\right].$

Proof:

- Let w_0 be a positive solution of Eq. (E_θ) such that $k(w_0) = \beta(\theta)$Then

$$\int_\Omega h(x) w_0^{r+1} dx = \int_\theta h(x) w_0^{r+1} dx > 0.$$

Set $t_\lambda = t^+(w_0)$ as defined by Lemma (2.4)

Hence $t_\lambda w_0 \in M_\lambda^+$ and

$$J_\lambda(t_\lambda w_0) = \frac{t_\lambda^p}{p}\|w_0\|_X^p - \frac{t_\lambda^{s+1}}{s+1}\int_\Omega g(x)|w_0|^{s+1}dx$$
$$-\frac{\lambda t_\lambda^{r+1}}{r+1}\int_\Omega h(x)|w_0|^{r+1}dx = \left(\frac{1}{p} - \frac{1}{r+1}\right) t_\lambda^p \|w_0\|_X^p$$
$$+\left(\frac{1}{r+1} - \frac{1}{s+1}\right) t_\lambda^{s+1} \int_\Omega g(x)|w_0|^{s+1} dx$$
$$< -\frac{p-r-1}{r+1} t_\lambda^p \beta(\theta) < 0.$$

This yields

$$\alpha_\lambda \le \alpha_\lambda^+ < -\frac{p-r-1}{r+1} t_\lambda^p \beta(\theta) < 0.$$

For $u \in M_\lambda$ we have

$$\|u\|_X^p = \int_\Omega g(x)|u|^{s+1}dx + \lambda\int_\Omega h(x)|u|^{r+1}dx.$$

Then using the Holder an Young inequalities

$$J_\lambda(u) = \frac{s+1-p}{p(s+1)}\|u\|_X^p$$
$$-\lambda\left(\frac{s-r}{(r+1)(s+1)}\right)\int_\Omega h(x)|u|^{r+1}dx$$
$$\ge \frac{s+1-p}{p(s+1)}\|u\|_X^p$$
$$-\lambda\left(\frac{s-r}{(r+1)(s+1)}\right)\|h\|_{L^{r_0}} \frac{1}{S^{\frac{r+1}{p}}}\|u\|_X^{r+1}$$
$$\ge \left[\left(\frac{s+1-p}{p(s+1)}\right) - \lambda\left(\frac{s-r}{p(s+1)}\right)\right]\|u\|_X^p$$
$$-\lambda\left(\frac{(s-r)(p-r-1)}{p(s+1)(r+1)}\right)\left(\|h\|_{L^{r_0}}\frac{1}{S^{\frac{r+1}{p}}}\right)^{\frac{p}{p-r-1}}$$
$$= \frac{1}{p(s+1)}\left[(s+1-p) - \lambda(s-r)\right]\|u\|_X^p$$
$$-\lambda\left(\frac{(s-r)(p-r-1)}{p(s+1)(r+1)}\right)\left(\|h\|_{L^{r_0}}\frac{1}{S^{\frac{r+1}{p}}}\right)^{\frac{p}{p-r-1}}.$$

Thus, J_λ is coercive on M_λ and

$$J_\lambda(u) \ge -\lambda\left(\frac{(s-r)(p-r-1)}{p(s+1)(r+1)}\right)\left(\|h\|_{L^{r_0}}\frac{1}{S^{\frac{r+1}{p}}}\right)^{\frac{p}{p-r-1}}$$

for all $\lambda \in \left(0, \frac{s+1-p}{s-r}\right].$

Proof of Theorem 1.1.

Theorem 3.1: Let $\lambda_0 = \min\left\{\lambda_1, \lambda_2, \frac{s+1-p}{s-r}\right\}$ for each λ $\in (0, \lambda_0)$

There exists a minimizing sequence $\{u_n\} \subset M_\lambda$ such that

$$J_\lambda(u_n) = \alpha_\lambda + 0(1) \quad , \quad J'_\lambda(u_n) = 0(1) \quad \text{in} \quad X^{-1}$$

There exists a minimizing sequence $\{u_n\} \subset M_\lambda^-$ such that

$$J_\lambda(u_n) = \alpha_\lambda^- + 0(1) \quad , \quad J'_\lambda(u_n) = 0(1) \quad \text{in} \quad X^{-1}$$

Proof: The proof is almost the same as that in [3, Proposition 9] and we omit that.

Now we establish the existence of a local minimum for J_λ on M_λ^+.

Theorem 3.2: Let $\lambda_0 > 0$ as in Proposition 3.1, then for $\lambda \in (0, \lambda_0)$ the functional J_λ has a minimizer u_0^+ in M_λ^+ and it satisfies

(I) $J_\lambda(u_0^+) = \alpha_\lambda = \alpha_\lambda^+$

(ii) u_0^+ is a positive solution of Eq. (E_λ)

(iii) $J_\lambda(u_0^+) \to 0$ as $\lambda \to 0$

Proof: Let $\{u_n\}$ be a minimizing sequence J_λ on M_λ such that

$$J_\lambda(u_n) = \alpha_\lambda + 0(1), J'_\lambda(u_n) = 0(1) \; in \; X^{-1}$$

then by Lemma (2.6) and the compact imbedding theorem, there exist a subsequence $\{u_n\}$ and $u_0^+ \in X$ such that

$u_n \to u_0^+$ weakly in X

$u_n \to u_0^+$ strongly in L^{s+1}

$u_n \to u_0^+$ strongly in $L^{s+1}(\Omega)$

First, we claim that $\int_\Omega h(x)\left|u_0^+\right|^{r+1} dx \neq 0.$ If not, by (9) we conclude that

$$\int_\Omega h(x)\left|u_0^+\right|^{r+1} dx = 0$$

and

$$\int_\Omega h(x)|u_n|^{r+1} dx \to 0 \; as \; n \to \infty. \; u_n \to u_0^+$$

Thus

$$\int_\Omega |\nabla u|^p \, dx = \int_\Omega g(x)|u|^{s+1} dx + 0(1),$$

and

$$\begin{aligned} J_\lambda(u_n) &= \frac{1}{p}\int_\Omega |\nabla u_n|^p \, dx - \frac{1}{s+1}\int_\Omega g(x)|u_n|^{s+1} dx \\ &\quad - \frac{\lambda}{r+1}\int_\Omega h(x)|u_n|^{r+1} dx \\ &= \left(\frac{1}{p} - \frac{1}{s+1}\right)\int_\Omega g(x)|u_n|^{s+1} dx + 0(1) \\ &= \left(\frac{1}{p} - \frac{1}{s+1}\right)\int_\Omega g(x)\left|u_0^+\right|^{s+1} dx \quad as \; n \to \infty, \end{aligned}$$

this contradicts $J_\lambda(u_n) \to \alpha_\lambda < 0$ as $n \to \infty$ In particular, $u_0^+ \in M_\lambda$ is a nonzero solution of Eq. (E_λ) and $J_\lambda(u_0^+) \geq \alpha_\lambda.$ We now prove that $u_n \to u_0^+$ strongly in X. Supposing the contrary, then $\left\|u_0^+\right\|_X < \liminf_{n\to\infty}\|u_n\|_X$ and so

$$\begin{aligned} &\left\|u_0^+\right\|_X^p - \int_\Omega g(x)\left|u_0^+\right|^{s+1} dx - \lambda\int_\Omega h(x)\left|u_0^+\right|^{r+1} dx \\ &< \liminf_{n\to\infty}\left(\|u_n\|_X^p - \int_\Omega g(x)|u_n|^{s+1} dx \right. \\ &\left. -\lambda\int_\Omega h(x)|u_n|^{r+1} dx\right) = 0, \end{aligned}$$

this contradicts with $u_0^+ \in M_\lambda$. Hence strongly in X. This implies

$$J_\lambda(u_n) \to J_\lambda(u_0^+) = \alpha_\lambda \; as \; n \to \infty.$$

Moreover, we have $u_0^- \in M_\lambda^-$, . In fact, if $u_0^- \in M_\lambda^-$, by Lemma (2.4) there exists unique t_0^+ and t_0^- such that $t_0^+ u_0^+ \in M_\lambda^+$ and $t_0^- u_0^+ \in M_\lambda^-$, we have $t_0^+ < t_0^- = 1.$ Since0

$$\frac{d}{dt}J_\lambda(t_0^+ u_0^+) = 0 \text{ and } \frac{d^2}{dt^2}J_\lambda(t_0^+ u_0^+) > 0,$$

there exists $t_0^+ < t^- \leq t_0^-$ such that

$$J_\lambda(t_0^+ u_0^+) < J_\lambda(t^- u_0^+).$$

By Lemma (2.4) we conclude that

$$J_\lambda\left(t_0^+u_0^+\right) < J_\lambda\left(t^-u_0^+\right) \le J_\lambda\left(t_0^-u_0^+\right) = J_\lambda\left(u_0^+\right),$$

Which is a contradiction. Since $J_\lambda\left(u_0^+\right) = J_\lambda\left(\left|u_0^+\right|\right)$ and $\left|u_0^+\right| \in M_\lambda^+(\Omega)$, and using Lemma (2.2) we may assume that u_0^+ is nonnegative solution. By [4, Lemma(2.1)] we have $u_0^+ \in L^\infty(\Omega)$. Then we can apply the Harnack inequality [5] in order to get that u_0^+ is positive in Ω Moreover, by Lemma (2.6), $J_\lambda\left(u_0^+\right) < 0$ and

$$J_\lambda\left(u_0^+\right) \ge -\lambda\left(\frac{(s-r)(p-r-1)}{p(s+1)(r+1)}\right)\left(\|h\|_{L^{r_0}} \frac{1}{S^{\frac{r+1}{p}}}\right)^{\frac{p}{p-r-1}}.$$

We obtain $J_\lambda\left(u_0^+\right) \to 0$ as $\lambda \to 0$

Next, we establish the existence of a local minimum for J_λ on M_λ^-.

Theorem 3.3: Let $\lambda_0 > 0$ as in Proposition 3.1, then for $\lambda \in (0,\lambda_0)$ the functional J_λ has a minimizer u_0 in M_λ^- nd it satisfies

(I) $J_\lambda\left(u_0^-\right) = \alpha_\lambda^-$

(ii) u_0^- is a positive solution of Eq.(E_λ).

Proof: By Proposition 3.1 (ii), there exists a minimizing sequence $\{u_n\}$ for J_λ on M_λ^- such that

$J_\lambda(u_0) = \alpha_\lambda^- + 0(1)$ and $J'_\lambda(u_n) = 0(1)$ in X^{-1}.

By Lemma (2.1) and the compact imbedding theorem, there exist a subsequence $\{u_n\}$ and $u_0^- \in M_\lambda^-$ such that

$u_n \to u_0^-$ weakly in X

$u_n \to u_0^-$ strongly in L^{s+1}

$u_n \to u_0^-$ strongly in $L^{r+1}(\Omega)$

We now prove that $u_n \to u_0^-$ strongly in X.

Suppose otherwise, then $\left\|u_0^-\right\|_X < \liminf_{n\to\infty}\|u_n\|_X$ and so

$$\begin{aligned}&\left\|u_0^-\right\|_X^p - \int_\Omega g(x)\left|u_0^-\right|^{s+1}dx - \lambda\int_\Omega h(x)\left|u_0^-\right|^{r+1}dx\\ &< \liminf_{n\to\infty}\Big(\|u_n\|_X^p - \int_\Omega g(x)|u_n|^{s+1}dx\\ &\quad -\lambda\int_\Omega h(x)|u_n|^{r+1}dx\Big) = 0.\end{aligned}$$

This contradicts with $u_0^- \in M_\lambda^-$. Hence $u_n \to u_0^-$ strongly in X.

This implies

$$J_\lambda(u_n) \to J_\lambda\left(u_0^-\right) = \alpha_\lambda^- \ as\ n \to \infty.$$

Since $J_\lambda\left(u_0^-\right) = J_\lambda\left(\left|u_0^-\right|\right)$ and $\left|u_0^-\right| \in M_\lambda^-(\Omega)$ by Lemma (2.2) we may assume that u_0^- is a nonnegative solution, $u_0^- \ge 0$ in Ω and $u_0^- \ne 0$.

Now, we complete the proof of Theorem (1.1). By Theorems (3.2), (3.3), for Eq.(E_λ) there exist two positive solutions u_0^+ and u_0^- such that

$$u_0^+ \in M_\lambda^+, u_0^- \in M_\lambda^-.$$

Since $M_\lambda^+ \cap M_\lambda^- = \varnothing$. This implies that u_0^+ and u_0^- are different.

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