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Characterization Tangent Developable of Null Biharmonic Curves in the Lorentzian Heisenberg Group Heis^3

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Abstract: In this paper, we study in particular developable surfaces, a special type of ruled surface in the Lorentzian Heisenberg group Heis^3. We give necessary and sufficient conditions for null biharmonic curves in the Lorentzian Heisenberg group Heis^3. We find out explicit parametric equations of tangent developable of biharmonic curve in the Lorentzian Heisenberg group Heis^3. Mathematics Subject Classification (2010). 58E20.

Key words: Heisenberg group · Biharmonic curve · Null curve · Helices · Developable surface

INTRODUCTION

Developable surfaces are especially important to the home boatbuilder because they are often working with sheet materials like plywood, steel or aluminum. Developable surfaces can be formed from flat sheets without stretching, so the forces required to form sheet materials into developable surfaces are much less than for other surfaces. In some cases, particularly with plywood, the force required to form non-developable surfaces could be so large that the material is damaged internally when it is formed. Another advantage of developable surfaces is that the development, or flattened out shape, of such a surface is exact. When other types of surfaces are expanded (note the difference in terms - " expansion" is flattening out a non-developable surface) the shape of the expansion depends on the distortion field applied to form it, so that there is no single exact expansion without detailed forming information. If you are either designing or lofting plywood or metal hulls you should understand developable surfaces and the methods for working with them.

On the other hand, harmonic maps $f:(M, g)\to(N, h)$ between Riemannian manifolds are the critical points of the energy

$$E(f)=\frac{1}{2}\int_M |df|^2 v_g, \tag{1.1}$$

and they are therefore the solutions of the corresponding Euler--Lagrange equation. This equation is given by the vanishing of the tension field

$$\tau(f)=\operatorname{trace}\nabla df. \tag{1.2}$$

As suggested by Eells and Sampson in [8], we can define the bienergy of a map f by

$$E_2(f)=\frac{1}{2}\int_M |\tau(f)|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [10,11], showing that the Euler--Lagrange equation associated to E_2 is

$$\begin{aligned}\tau_2(f)&=-\mathrm{J}^f(\tau(f))=-\Delta\tau(f)\\ &-\operatorname{trace}R^N(df,\tau(f))df=0,\end{aligned} \tag{1.4}$$

Where J^f is the Jacobi operator of f. The equation $\tau_2(f)=0$ is called the biharmonic equation. Since J^f is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

In this paper, we study in particular developable surfaces, a special type of ruled surface in the Lorentzian Heisenberg group Heis^3. We give necessary and sufficient conditions for null biharmonic curves in the Lorentzian Heisenberg group Heis^3. We find out explicit parametric equations of tangent developable of biharmonic curve in the Lorentzian Heisenberg group Heis^3

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The Lorentzian Heisenberg Group Heis^3: The Lorentzian Heisenberg group Heis^3 can be seen as the space R ^{3} endowed with the following multiplication:

$$(\bar{x},\bar{y},\bar{z})(x,y,z) = (\bar{x}+x,\bar{y}+y,\bar{z}+z-\bar{x}y+x\bar{y}).$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Lorentz metric g is given by:

$$g = -dx^2 + dy^2 + (xdy + dz)^2,$$

Where

$$\omega^1 = dz + xdy, \quad \omega^2 = dy, \quad \omega^3 = dx$$

is the left-invariant orthonormal coframe associated with the orthonormal left-invariant frame,

$$\mathbf{e}_1 = \frac{\partial}{\partial z}, \mathbf{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \mathbf{e}_3 = \frac{\partial}{\partial x} \tag{2.1}$$

for which we have the Lie products

$$\mathbf{e}_2,\mathbf{e}_3] = 2\mathbf{e}_1, [\mathbf{e}_3,\mathbf{e}_1] = 0, [\mathbf{e}_2,\mathbf{e}_1] = 0,$$

With

$$g(\mathbf{e}_1,\mathbf{e}_1) = g(\mathbf{e}_2,\mathbf{e}_2) = 1, \; g(\mathbf{e}_3,\mathbf{e}_3) = -1. \tag{2.2}$$

Proposition 2.1: For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:

$$\nabla = \begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix}, \tag{2.3}$$

Where the (i, j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ *for our basis*

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}.$$

We adopt the following notation and sign convention for Riemannian curvature operator:

$$R(X,Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{X,Y]}Z.$$

The Riemannian curvature tensor is given by

$$R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

Moreover we put

$$R_{abc} = R(\mathbf{e}_a,\mathbf{e}_b)\mathbf{e}_c, \; R_{abcd} = R(\mathbf{e}_a,\mathbf{e}_b,\mathbf{e}_c,\mathbf{e}_d),$$

ewhere the indices a, b, c and d take the values 1,2 and 3.

$$R_{121} = -\mathbf{e}_2, R_{131} = -\mathbf{e}_3, \; R_{232} = 3\mathbf{e}_3$$

and

$$R_{1212} = -1, R_{1313} = 1, R_{2323} = -3. \tag{2.4}$$

3 Null Biharmonic Curves in the Lorentzian Heisenberg Group Heis^3: Let $\gamma : I \to Heis^3$ be a null curve on the Lorentzian Heisenberg group Heis^3 parametrized by arc length. Let {T, N, B} be the Frenet frame fields tangent to the Lorentzian Heisenberg group Heis^3 along γ defined as follows:

T is the unit vector field γ' tangent to γ, N is the unit vector field in the direction of ∇_TT (normal to γ) and B is chosen so that {T, N B} is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa_1\mathbf{N} \\ \nabla_{\mathbf{T}}\mathbf{N} &= \kappa_2\mathbf{T} - \kappa_1\mathbf{B} \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\kappa_2\mathbf{N}, \end{aligned} \tag{3.1}$$

Where

$$\begin{aligned} g(\mathrm{T,T}) = g(\mathrm{B,B}) = 0, g(\mathrm{N,N}) = 1 \\ g(\mathrm{T,N}) = g(\mathrm{N,B}) = 0, g(\mathrm{T,B}) = 1 \end{aligned} \tag{3.2}$$

and k_1 is the curvature of γ and k_2 is its torsion. With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, we can write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, 3.3 \\ \mathbf{B} &= \mathbf{T}\times\mathbf{N} = B_1\mathbf{e}_1 + B_2 e_2 + B_3\mathbf{e}_3. \end{aligned} \tag{3.3}$$

Theorem 3.2. Let $\gamma : I \to Hies^3$ be a non-geodesic null curve parametrized by arc length. γ is a non-geodesic null biharmonic curve if and only if

$$\begin{aligned} \kappa_1 &= \text{constant} \neq 0, \\ k_1 k_2 &= -2B_1^2 \\ \kappa_2' &= 4N_1B_1. \end{aligned} \tag{3.4}$$

Corollary 3.3. Let $\gamma : I \to Heis^3$ be a null curve with constant curvature and $N_1B_1 \neq 0$. Then γ is not biharmonic [21].

Corollary 3.4. $\gamma : I \to Heis^3$ is null biharmonic if and only if

$$\begin{aligned} \kappa_1 &= \text{constant} \neq 0, \\ k_2 &= \text{constant} \\ N_1B_1 &= 0, \\ \kappa_1\kappa_2 &= -2B_1^2. \end{aligned} \tag{3.5}$$

Corollary 3.6. *If* $N_1 = 0$,, then

$$\mathbf{T}(s) = \sinh \Psi_0 \mathbf{e}_1 + \sinh \Psi_0 \sinh \Phi (s)\mathbf{e}_2 + \sinh \Psi_0 \cosh \Phi(s)\mathbf{e}_3, \tag{3.6}$$

Where $\Psi_0 \in$�.

Proof: Using Corollary 3.4, we obtain (3.6) and corollary is proved.

4 Tangent Developable of Null Biharmonic Curve in the Lorentzian Heisenberg Group Heis³: Ruled surfaces are swept out by the motion of a straight line in space. More formally, the image of the map $\hat{O}_{(\gamma,\delta)} : I \times R \to Heis_3$ defined by

$$\hat{O}_{(\gamma,\delta)}(s,u) = \gamma(s) + u\delta(s),\ (s,u) = 0$$

is called a ruled surface in $Heis^3$ where $\gamma : I \to Heis^3$ $\delta : I \to Heis^3$ are smooth mappings and I is an open interval or a unit circle S^1.

We call γ the base curve and δ the director curve. The straight lines $u \to \gamma(s) + u\delta(s)$ are called rulings.

Note that we allow our ruled surfaces to possess singular points, that is points at which the partial derivatives of $\hat{O}_{(\gamma,\delta)}$ are linearly indepedent, i.e. which satisfy

$$\hat{O}_s(s,u) \times \hat{O}_u(s,u) = 0$$
$$\Leftrightarrow \left(\gamma'(s) + u\delta'(s)\right) \times \delta(s) = 0$$
$$\Leftrightarrow \gamma'(s) \times \delta(s) + u\delta'(s) \times \delta(s) = 0.$$

We now consider a special type of ruled surface, which has been studied for over a century, the developable surface. Informally, these are surfaces which can be attened onto a plane without distortion, so are a transformation (e.g. folding or bending) of a plane in $Heis^3$. It is this fundamental property which has long ensured their useful application in engineering and manufacturing. More recently, their use has spread to the computer sciences, in computer-aided design; their isometric properties make them ideal primitives for texture mapping.

Definition 4.1. A smooth surface $\hat{O}_{(\gamma,\delta)}$ is called a developable surface if its Gaussian curvature K vanishes everywhere on the surface.

Proposition 4.2. A ruled surface is a developable surface [4] if:

$$\left(\gamma'(s) \times \delta(s)\right) \times \delta'(s) = 0. \tag{4.1}$$

We can give a geometric interpretation of Proposition 4.2 by computing the Gaussian curvature at a regular point. Since

$$\hat{O}_{tt} = \gamma'' + u\delta'', \hat{O}_{tu}\delta' = \delta', \hat{O}_{uu} = 0 \tag{4.2}$$

Computations of the coeffecients of the second fundamental form give:

$$\begin{aligned} N &= 0, \\ M &= \mathbf{n}.\text{�}_{tu} b \\ &= \frac{(\hat{O}_t \times \hat{O}_u).\hat{O}_{tu}}{|\hat{O}_t \times \hat{O}_u|} \\ &= \frac{(\gamma' \times \delta).\delta'}{|\text{�}_t \times \text{�}_u|^2} \end{aligned} \tag{4.3}$$

$$\hat{O}_{(\gamma,\delta)}(s,u) = \gamma(s) = \gamma(s) + u\gamma(s) \tag{4.4}$$

The tangent developable is the envelope of the family of osculating planes along γ, where the osculating plane at $\gamma(s)$ is the plane generated by the tangent vector $\gamma'(s)$ and the principal normal $N(s)$.

Theorem 4.3. $\gamma : I \to Heis^3$ be a non-geodesic null curve on Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. Then, the parametric equations for tangent developable of γ are

$$\begin{aligned} x(s,u) &= \frac{1}{(a - 2\sinh \Psi_0)} \sinh \Psi_0 \sinh((a - 2\sinh \Psi_0)s + \sigma) \\ &+ u \sinh \Psi_0 \cosh[(a - 2\sinh \Psi_0)s + \sigma] + a_1, \\ y(s,u) &= \frac{1}{(a - 2\sinh \Psi_0)} \sinh \Psi_0 \cosh((a - 2\sinh \Psi_0)s + \sigma) \\ &+ u \sinh \Psi_0 \sinh[(a - 2\sinh \Psi_0)s + \sigma] + a_2, \\ z(s,u) &= \left[\sinh \Psi_0 - \frac{[\sinh \Psi_0]^2}{(a - 2\sinh \Psi_0)}\right] s \\ &- \frac{[\sinh \Psi_0]^2}{2(a - 2\sinh \Psi_0)^2} \sinh 2((a - 2\sinh \Psi_0)s + \sigma) \\ &- \frac{c_1}{(a - 2\sinh \Psi_0)} \sinh \Psi_0 \cosh((a - 2\sinh \Psi_0)s + \sigma) \\ &+ u \sinh \Psi_0 \cosh[(a - 2\sinh \Psi_0)s + \sigma] \\ &- \frac{u \sinh \Psi_0}{(a - 2\sinh \Psi_0)} \sinh^2[(a - 2\sinh \Psi_0)s + \sigma] \\ &- \frac{a_1 \sinh \Psi_0}{(a - 2\sinh \Psi_0)} \sinh[(a - 2\sinh \Psi_0)s + \sigma] + a_3, \end{aligned} \tag{4.5}$$

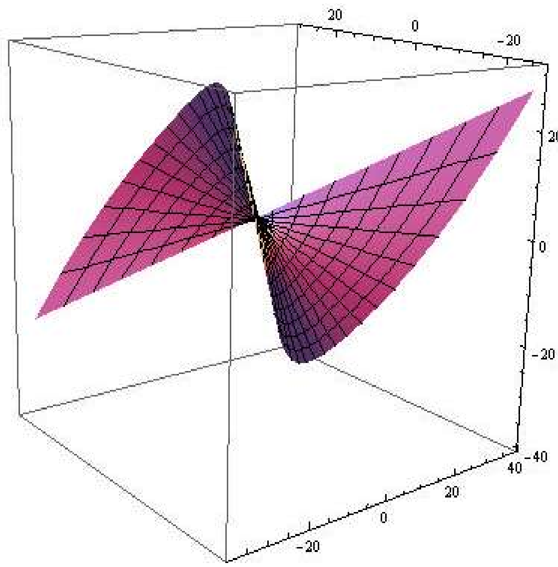

Fig. 1:

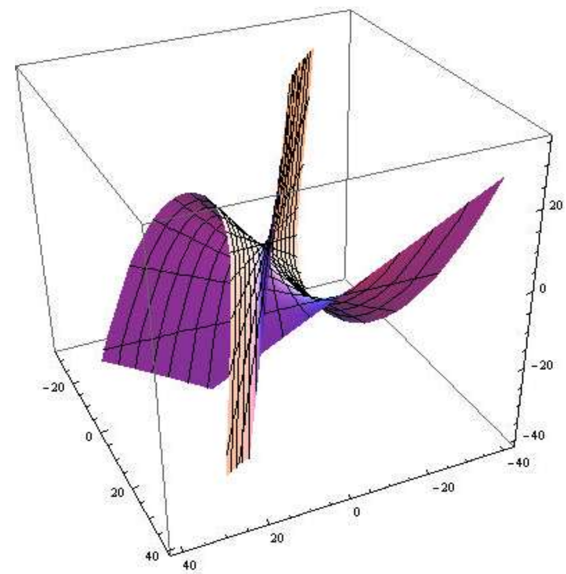

Fig. 2:

Where a_1, a_2, a_3 is constant of integration and $a = \pm\sqrt{4\sinh\Psi_0 + \frac{\kappa_1}{\sinh^3\Psi_0}}$.

Proof: The covariant derivative of the vector field T is:

$$\nabla_{\mathbf{T}}\mathbf{T} = T_1'\mathbf{e}_1 + (T_2' + 2T_1T_3)\mathbf{e}_2 + (T_3' + 2T_1T_2)\mathbf{e}_3. \quad (4.6)$$

From (3.6), we have

$$\nabla_{\mathbf{T}}\mathbf{T} = (\Phi'(s)\sinh\Psi_0\cosh\Phi(s) + 2\sinh^2\Psi_0\cosh\Phi(s))e_2 + (\Phi'(s)\sinh\Psi_0\sinh\Phi(s) + 2\sinh^2\Psi_0\sinh\Phi(s))e_3.$$

Since $|\nabla_{\mathbf{T}}\mathbf{T}| = \kappa_1$, we obtain

$$\Phi(s) = (a - 2\sinh\Psi_0)s + \sigma, \quad (4.7)$$

Where $\sigma \in$ R and $a = \pm\sqrt{4\sinh\Psi_0 + \frac{\kappa_1}{\sinh^3\Psi_0}}$.

Using (2.1), we get

$$\frac{\partial}{\partial x} = \mathbf{e}_3, \frac{\partial}{\partial y} = \mathbf{e}_2 + x\mathbf{e}_3, \frac{\partial}{\partial z} = \mathbf{e}_1.$$

Therefore, we easily have:

$$\begin{aligned}
\frac{dx}{ds} &= \sinh\Psi_0\cosh[(a - 2\sinh\Psi_0)s + \sigma],\\
\frac{dy}{ds} &= \sinh\Psi_0\sinh[(a - 2\sinh\Psi_0)s + \sigma],\\
\frac{dz}{ds} &= \sinh\Psi_0\cosh[(a - 2\sinh\Psi_0)s + \sigma]\\
&-\frac{\sinh\Psi_0}{(a - 2\sinh\Psi_0)}\sinh^2[(a - 2\sinh\Psi_0)s + \sigma]\\
&-\frac{a_1\sinh\Psi_0}{(a - 2\sinh\Psi_0)}\sinh[(a - 2\sinh\Psi_0)s + \sigma].
\end{aligned} \quad (4.8)$$

By direct computations, substituting (4.8) in (4.4), we get (4.5).

If we use Mathematica in Theorem 4.3 for different constant, yields

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